Answer on Question #47342 – Math – Abstract Algebra

(a) Show that <x > is not a maximal ideal in z[x].

(b) List all the subgroups of Z18, along with 3 their generators.

(c) Let H=< (1 2) > and k = < (1 2 3) > be subgroups of S_3. Show that S_3 = Hk. Is S_3 an internal direct product of H and k? Justify your answer.

(d) Check whether or not { (2, 5), (1, 3), (5, 2), (3, 1) is an equivalence relation on { 1, 2, 3, 5 }.

Solution:

We know that [x] is just all polynomials with even coefficients. This is not maximal because Z[x] contains many ideals more than Z. For example, it contains the ideal <x>, which is all things that have positive degree (i.e non-constants).

In the case of Z[x], the ideal equal to

$$(x) = \{a_1x + a_2x^2 + \ldots a^n x^n;\ x_k \in Z, 1 \le k \le n\}$$

Is a principal ideal of Z[x] generated x. $\frac{Z[x]}{(x)} \cong Z$, an integral domain. So, (x) is a prime ideal of Z[x]. Let J = (x,2) the ideal generated by x and 2 in Z[x]. J is a maximal ideal (of Z[x]), which contain polynomials of the form:

$$a_0 + a_1x + a_2x^2 + \ldots a^n x^n;\ a_i \in Z, 0 \le i \le n$$

Where a_0 is even. As $\frac{Z[x]}{J}$ is a field having two elements, J is a maximum ideal of Z[x] containing the prime ideal (x). So, the prime ideal (x) is not a maximum ideal of Z[x]. Thus, Z[x] is not a Dedekind domain. Z[x] is not a PID also, as a principal ideal domain has to be a Dedekind domain. This conclusion is also obvious from the fact that J is not a principal ideal of Z[x].

(b) List all the subgroups of Z_{18}, along with 3 their generators.

The divisors of 18 are 1, 2, 3, 6, 9, and 12. There is a subgroup of each of these orders, and they are generated by [0], [9], [6], [3], [2], and [1] respectively.

That is, the subgroups of Z_{18} $\langle[0]\rangle$, $\langle[9]\rangle$, $\langle[6]\rangle$, $\langle[3]\rangle$, $\langle[2]\rangle$ and $\langle[1]\rangle$.

(c) Let H=< (1 2) > and k = < (1 2 3) > be subgroups of S_3. Show that S_3 = Hk. Is S_3 an internal direct product of H and k? Justify your answer.

We write e for the identity element of S_3, note that H = {e, (1,2)} and K = {e, (1,2,3), (1,3,2)}. So Hk clearly contains e, (1,2), (1,2,3), and (1,3,2). The computations

(1,2)(1,2,3) = (2,3)

(1,2)(1,3,2) = (1,3)

We need to show that HK also contains (2,3) and (1,3). Since S_3 = {e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)} we have shown that Hk contains every element of S_3, and hence S_3= Hk.

We note that H and k are both abelian groups (because e.g. they are both cyclic). A direct product of abelian groups (whether internal or external) is abelian. Since $S_3$is not abelian, this tells us that $S_3$is not an internal direct product of H and K. (We know that any group of order less than 6 is abelian, this argument shows more: $S_3$is not an internal direct product of any two of its proper subgroups.)

(d) Check whether or not { (2, 5), (1, 3), (5, 2), (3, 1) is an equivalence relation on { 1, 2, 3, 5 }.

In mathematics, an equivalence relation is the relation that holds between two elements if and only if they are members of the same cell within a set that has been partitioned into cells such that every element of the set is a member of one and only one cell of the partition. The intersection of any two different cells is empty; the union of all the cells equals the original set. These cells are formally called equivalence classes.

A relation R on a set A is an equivalence relation if and only if R is

• reflexive,

• symmetric, and

• transitive.

Let R be a relation on set A. Any equivalence relation must be reflexive i.e. for all a in A we must have (a, a) in R.

In our example we must have all (1, 1), (2, 2), (3, 3), (5, 5) present in R which we don't. In fact, none are in R.

Therefore Is not an equivalence relation, R is not reflexive, since (1, 1)$\notin$ R for example.