Answer on Question# #47136 – Mathematics – Calculus

Question:

Find the n-th derivative of

$$y(x) = \frac{x}{x^2+a^2}. \qquad (1)$$

Solution:

To find derivatives of the function (1) we use the Leibniz formula for higher-order derivatives:

$$(u(x)v(x))^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)}, \qquad (2)$$

where $C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. In this case assuming $u = x$ and $v = \frac{1}{x^2+a^2}$ we obtain

$$\left(\frac{x}{x^2+a^2}\right)^{(n)} = C_n^0 (x)^{(n)} (\frac{1}{x^2+a^2})^{(0)} + C_n^1 (x)^{(n-1)} (\frac{1}{x^2+a^2})^{(1)} + C_n^2 (x)^{(n-2)} (\frac{1}{x^2+a^2})^{(2)} + \cdots$$
$$+ C_n^{n-1} (x)^{(n-(n-1))} (\frac{1}{x^2+a^2})^{(n-1)} + C_n^n (x)^{(n-n)} (\frac{1}{x^2+a^2})^{(n)}$$
$$= C_n^0 (x)^{(n)} (\frac{1}{x^2+a^2})^{(0)} + C_n^1 (x)^{(n-1)} (\frac{1}{x^2+a^2})^{(1)} + C_n^2 (x)^{(n-2)} (\frac{1}{x^2+a^2})^{(2)} + \cdots$$
$$+ C_n^{n-1} (x)^{(1)} (\frac{1}{x^2+a^2})^{(n-1)} + C_n^n (x)^{(0)} (\frac{1}{x^2+a^2})^{(n)}.$$

Since

$$C_n^0 = \frac{n!}{0!\,(n-0)!} = 1, C_n^n = \frac{n!}{n!\,(n-n)!} = 1, C_n^{n-1} = \frac{n!}{(n-1)!\,(n-(n-1)!} = n$$

$$(x)^{(0)} = x, \qquad (x)' = 1, \ \ (x)'' = (x)''' = (x)^{(4)} = \cdots = (x)^{(n)} = 0,$$

we receive the following expression

$$\left(\frac{x}{x^2+a^2}\right)^{(n)} = C_n^{n-1}(x)'(\frac{1}{x^2+a^2})^{(n-1)} + C_n^n x^{(0)} (\frac{1}{x^2+a^2})^{(n)} = n(\frac{1}{x^2+a^2})^{(n-1)} + x(\frac{1}{x^2+a^2})^{(n)}. \qquad (3)$$

Now we need calculate the derivatives $(\frac{1}{x^2+a^2})^{(n)}$ and $(\frac{1}{x^2+a^2})^{(n-1)}$. Let`s rewrite the function $\frac{1}{x^2+a^2}$ in the most convenient form:

$$\frac{1}{x^2+a^2} = \frac{1}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right). \qquad (4)$$

Then we have

$$(\frac{1}{x^2+a^2})^{(n)} = \left(\frac{1}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)\right)^{(n)} = \frac{1}{2ai}\left(\left(\frac{1}{x-ai}\right)^{(n)} - \left(\frac{1}{x+ai}\right)^{(n)}\right). \qquad (5)$$

$$(\frac{1}{x^2+a^2})^{(n-1)} = \left(\frac{1}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)\right)^{(n-1)} = \frac{1}{2ai}\left(\left(\frac{1}{x-ai}\right)^{(n-1)} - \left(\frac{1}{x+ai}\right)^{(n-1)}\right). \qquad (5a)$$

As we know, the n-th (n-1) derivatives of linear fractional function $y = \frac{ax+b}{cx+d}$ are

$$y^{(n)} = \left(\frac{ax+b}{cx+d}\right)^{(n)} = (ad-bc)(-1)^{n-1}n!\,(cx+d)^{-(n+1)}c^{n-1}. \quad (6)$$

$$y^{(n-1)} = \left(\frac{ax+b}{cx+d}\right)^{(n-1)} = (ad-bc)(-1)^{(n-1)-1}(n-1)!\,(cx+d)^{-((n-1)+1)}c^{(n-1)-1}. \quad (6)$$

Hence we can write (putting in (6) *a=0, b=1, c=1, d=*$\pm$*ai)*:

$$\left(\frac{1}{x^2+a^2}\right)^{(n)} = \frac{1}{2ai}\left((-1)(-1)^{n-1}n!\,\big(x+(-ai)\big)^{-(n+1)}1^{n-1} - (-1)(-1)^{n-1}n!\,(x+ai))^{-(n+1)}1^{n-1}\right)$$

$$= \frac{1}{2ai}\left((-1)^n n!\,(x-ai)^{-(n+1)} - (-1)^n n!\,(x+ai))^{-(n+1)}\right)$$

$$= \frac{(-1)^n n!}{2ai}\left(\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right),$$

$$\left(\frac{1}{x^2+a^2}\right)^{(n-1)}$$

$$= \frac{1}{2ai}\left((-1)(-1)^{(n-1)-1}(n-1)!\,\big(x+(-ai)\big)^{-(n-1+1)}1^{n-1-1} - (-1)(-1)^{(n-1)-1}(n - 1)!\,(x+ai))^{-(n-1+1)}1^{n-1-1}\right)$$

$$= \frac{1}{2ai}\left((-1)^{n-1}(n-1)!\,\big(x+(-ai)\big)^{-n} - (-1)^{n-1}(n-1)!\,(x+ai))^{-n}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ai}\left(\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right).$$

Therefore we obtain

$$\left(\frac{x}{x^2+a^2}\right)^{(n)} = n\cdot\frac{(-1)^{n-1}(n-1)!}{2ai}\left(\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right) + x\cdot\frac{(-1)^n n!}{2ai}\left(\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right),\ n>0. \quad (7)$$

or

$$\left(\frac{x}{x^2+a^2}\right)^{(n)} = n\cdot\frac{(-1)^{n-1}(n-1)!}{2ai}\left(\frac{(x+ai)^n-(x-ai)^n}{(x^2+a^2)^n}\right) + x\cdot\frac{(-1)^n n!}{2ai}\left(\frac{(x+ai)^{n+1}-(x-ai)^{n+1}}{(x^2+a^2)^{n+1}}\right),\ n>0. \quad (7a)$$

Note that these formulas are valid only for *n>0.*

Let`s check out the final formulas for *n=0, 1, 2*

$$n=0:\ \left(\frac{x}{x^2+a^2}\right)^{(0)} = \left(\frac{x}{x^2+a^2}\right) = 0\cdot\left(\frac{1}{x^2+a^2}\right)^{(0-1)} + x\left(\frac{1}{x^2+a^2}\right) = \frac{x}{x^2+a^2}.$$

Here we use the formula (3).

n=1: $\left(\frac{x}{x^2+a^2}\right)^{(1)}=\left(\frac{x}{x^2+a^2}\right)'=1\cdot\frac{(-1)^{1-1}(1-1)!}{2ai}\left(\frac{1}{(x-ai)}-\frac{1}{(x+ai)}\right)+x\frac{(-1)^1 1!}{2ai}\left(\frac{1}{(x-ai)^{1+1}}-\frac{1}{(x+ai)^{1+1}}\right)=$
$\frac{1}{2ai}\left(\frac{1}{(x-ai)}-\frac{1}{(x+ai)}\right)-\frac{x}{2ai}\left(\frac{(x+ai)^2-(x-ai)^2}{(x^2+a^2)^2}\right)=\frac{1}{2ai}\left(\frac{x+ai-(x-ai)}{(x^2+a^2)}\right)-\frac{x}{2ai}\left(\frac{x^2+2aix-a^2-(x^2-2aix+a^2)}{(x^2+a^2)^2}\right)=$
$\frac{1}{(x^2+a^2)}-\frac{x}{2ai}\left(\frac{4aix}{(x^2+a^2)^2}\right)=\frac{1}{(x^2+a^2)}-\frac{2x^2}{(x^2+a^2)^2}=\frac{a^2-x^2}{(x^2+a^2)^2},$

n=2: $\left(\frac{x}{x^2+a^2}\right)^{(2)}=\left(\frac{x}{x^2+a^2}\right)''=2\frac{(-1)^{2-1}(2-1)!}{(2ai)}\left(\frac{1}{(x-ai)^2}-\frac{1}{(x+ai)^2}\right)+x\frac{(-1)^2 2!}{(2ai)}\left(\frac{1}{(x-ai)^{2+1}}-\frac{1}{(x+ai)^{2+1}}\right)=$
$-\frac{1}{ai}\left(\frac{(x+ai)^2-(x-ai)^2}{(x^2+a^2)^2}\right)+\frac{x}{ai}\left(\frac{(x+ai)^3-(x-ai)^3}{(x^2+a^2)^3}\right)=-\frac{1}{ai}\left(\frac{x^2+2aix-a^2-(x^2-2aix+a^2)}{(x^2+a^2)^2}\right)+$
$\frac{x}{ai}\left(\frac{x^3-3a^2x+3iax^2-ia^3-(x^3-3a^2x-3iax^2+ia^3)}{(x^2+a^2)^3}\right)=-\frac{1}{ai}\left(\frac{4aix}{(x^2+a^2)^2}\right)+\frac{x}{ai}\left(\frac{6iax^2-2ia^3}{(x^2+a^2)^3}\right)=-\left(\frac{4x}{(x^2+a^2)^2}\right)+$
$2x\left(\frac{3x^2-a^2}{(x^2+a^2)^3}\right)==-2x\frac{3a^2-x^2}{(x^2+a^2)^3}.$

The direct calculation of first two derivatives gives

$$y'=\left(\frac{x}{x^2+a^2}\right)'=\frac{1}{x^2+a^2}-\frac{2x^2}{(x^2+a^2)^2}=\frac{a^2-x^2}{(x^2+a^2)^2},$$

$$y''=\left(\frac{a^2-x^2}{(x^2+a^2)^2}\right)'=\frac{-4x}{(x^2+a^2)^2}+2x\frac{3x^2-a^2}{(x^2+a^2)^3}=-2x\frac{3a^2-x^2}{(x^2+a^2)^3},$$

As we see, the result of calculation is the same.

Answer. $\left(\frac{x}{x^2+a^2}\right)^{(n)}=n\cdot\frac{(-1)^{n-1}(n-1)!}{2ai}\left(\frac{(x+ai)^n-(x-ai)^n}{(x^2+a^2)^n}\right)+x\cdot\frac{(-1)^n n!}{2ai}\left(\frac{(x+ai)^{n+1}-(x-ai)^{n+1}}{(x^2+a^2)^{n+1}}\right),\ n>0.$