Answer on Question #47126 – Math – Calculus

The lemniscates r*2equal to a*2 cos2 thita revolves about a tangent at a pole . Show that the volume generated is pi square multiply a cube/4

Solution.

$$r^2 = a^2 cos2\theta$$

In cartesian coordinates:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

and the tangents in the origin are the lines $y = \pm x$. Let R be the region of the $x \geq 0$ halfplane bounded by the lemniscate. We just need to compute the area A of R, the centroid G of R, then apply the second Pappus' centroid theorem. By using polar coordinates we have:

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta = \frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cos2\theta d\theta = \frac{a^2}{2} * \frac{1}{2} sin2\theta \Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \frac{a^2}{2}$$

By symmetry, the centroid of R lies on the $y = 0$ line. Its abscissa is given by

$$G_x = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx} = \frac{\int_0^a xf(x)dx}{\frac{A}{2}} = \frac{4}{a^2}\int_0^a xf(x)dx =$$

$$= \frac{2\sqrt{2}}{a^2}\int_0^a x\sqrt{-2x^2 - a^2 + \sqrt{a^4 + 8a^2x^2}}\, dx = \frac{\pi a}{4\sqrt{2}}$$

hence the distance of the centroid of R from a tangent line in the origin is just $\frac{\pi a}{8}$, and $V = 2 * 2\pi * \frac{\pi a}{8} * A = \frac{\pi^2 a^3}{4}$.