Answer on Question #45826 – Math – Real Analysis

Problem.

Prove [0,1] is not countable without using the outer measure of an interval is its length?

Solution:

Suppose that [0,1] is countable.

If [0,1] is countable, then $[0,1] = (x_n)_{n\geq 0}$. We will construct the decreasing sequence of compact intervals. First we split [0,1] into three equal parts $\left[0,\frac{1}{3}\right]$, $\left[\frac{1}{3},\frac{2}{3}\right]$, $\left[\frac{2}{3},1\right]$. x_0 is not in one of this intervals. We denote this interval by $[a_0, b_0]$. Now we split $[a_0, b_0]$ into three equal parts I_1, I_2, I_3. x_1 is not in one of the given intervals. We denote this interval by $[a_1, b_1]$. In the same way we may construct interval $[a_n, b_n]$ for all nonnegative integer n. From the construction of the intervals we may notice that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $x_{n+1} \notin [a_{n+1}, b_{n+1}]$ and $b_{n+1} - a_{n+1} = \frac{1}{3}(b_n - a_n)$ for all nonnegative integer. $([a_n, b_n])$ is a decreasing sequence of compact intervals with $b_n - a_n \to 0$ (nested intervals), so by Nested Intervals Theorem they have the common point $P \in [0,1]$. If $[0,1] = (x_n)_{n\geq 0}$, then there exists m such that $x_m = P$, but $x\_m \notin [a_m, b_m]$ and therefore it cannot be in the intersection of all intervals, contradiction. Hence [0,1] isn't countable.