Answer on Question #44585 – Math – Abstract Algebra

Factorise 10 in two ways in $\mathbb{Z}[\sqrt{-6}]$. Hence, show that $\mathbb{Z}[\sqrt{-6}]$ is not a UFD.

Solution.

Note that:

$$10 = 4 - (-6) = 2^2 - \left(\sqrt{-6}\right)^2 = \left(2 + \sqrt{-6}\right)\left(2 - \sqrt{-6}\right);$$

So:

$$10 = 2 \cdot 5 = \left(2 + \sqrt{-6}\right)\left(2 - \sqrt{-6}\right);$$

Consider a norm of a number $x \in \mathbb{Z}\left[\sqrt{-6}\right]$:

$$x = a + b\sqrt{-6}, a, b \in \mathbb{Z} \Longrightarrow N(x) = a^2 + 6b^2;$$

$$\forall x, y \in \mathbb{Z}\left[\sqrt{-6}\right]: N(xy) = N(x)N(y);$$

Prove that $2, 5, 2 + \sqrt{-6}, 2 - \sqrt{-6}$ are irreducible elements of $\mathbb{Z}\left[\sqrt{-6}\right]$.

1) 2:

$$a, b \neq 1, 2 = ab \Longrightarrow 4 = N(2) = N(a)N(b) \Longrightarrow N(a) = N(b) = 2;$$

$$a = m + n\sqrt{-6}, N(a) = 2 \Longrightarrow m^2 + 6n^2 = 2 \Longrightarrow n = 0, m^2 = 2 - \text{contradiction } (m \in \mathbb{Z});$$

So, 2 is irreducible.

2) 5:

$$a, b \neq 1, 5 = ab \Longrightarrow 25 = N(5) = N(a)N(b) \Longrightarrow N(a) = N(b) = 5;$$

$$a = m + n\sqrt{-6}, N(a) = 5 \Longrightarrow m^2 + 6n^2 = 5 \Longrightarrow n = 0, m^2 = 5 - \text{contradiction } (m \in \mathbb{Z});$$

So, 5 is irreducible.

3) $2 + \sqrt{-6}$:

$$a, b \neq 1, 2 + \sqrt{-6} = ab \Longrightarrow 10 = N\left(2 + \sqrt{-6}\right) = N(a)N(b) \Longrightarrow N(a) = 2, N(b) = 5;$$

We proved in (a),(b) that $\forall a \in \mathbb{Z}\left[\sqrt{-6}\right]: N(a) \neq 2, N(a) \neq 5$.

So, $2 + \sqrt{-6}$ is irreducible.

4) $2 - \sqrt{-6}$:

$$a, b \neq 1, 2 - \sqrt{-6} = ab \Longrightarrow 10 = N\left(2 - \sqrt{-6}\right) = N(a)N(b) \Longrightarrow N(a) = 2, N(b) = 5;$$

We proved in (a),(b) that $\forall a \in \mathbb{Z}\left[\sqrt{-6}\right]: N(a) \neq 2, N(a) \neq 5$.

So, $2 - \sqrt{-6}$ is irreducible.

Now prove that $\mathbb{Z}\left[\sqrt{-6}\right]$ is not a UFD.

Assume the contrary. We have two distinct factorizations of a number 10. So, all irreducible factors are pairwise associated. Hence:

$$\left[\begin{array}{l}2\sim 2+\sqrt{-6}\\ 2\sim 2-\sqrt{-6}\end{array}\right. \Rightarrow \left[\begin{array}{l}N(2)=N\left(2+\sqrt{-6}\right)\\ N(2)=N\left(2-\sqrt{-6}\right)\end{array}\right. \Rightarrow 4=10-\text{contradiction.}$$

So, our assumption doesn't hold and $\mathbb{Z}\left[\sqrt{-6}\right]$ is not a UFD.