Answer on Question #40386 – Math – Differential Calculus

Solve: $xy' - y = e^{y'}$. Also obtain its singular solution.

Solution:

Solve the Clairaut equation $x\frac{dy(x)}{dx} - y(x) = e^{\frac{dy(x)}{dx}}$:

Subtract $x\frac{dy(x)}{dx}$ from both sides and divide by -1:

$$y(x) = -e^{\frac{dy(x)}{dx}} + x\frac{dy(x)}{dx}$$

Differentiate both sides with respect to x:

$$\frac{dy(x)}{dx} = \frac{dy(x)}{dx} - e^{\frac{dy(x)}{dx}}\frac{d^2y(x)}{dx^2} + x\frac{d^2y(x)}{dx^2}$$

Factor:

$$\frac{dy(x)}{dx} = \frac{dy(x)}{dx} + \frac{d^2y(x)}{dx^2}\left(-e^{\frac{dy(x)}{dx}} + x\right)$$

Subtract $\frac{dy(x)}{dx}$ from both sides:

$$\frac{d^2y(x)}{dx^2}\left(-e^{\frac{dy(x)}{dx}} + x\right) = 0$$

Solve $\frac{d^2y(x)}{dx^2} = 0$ and $x - e^{\frac{dy(x)}{dx}} = 0$ separately:

For $\frac{d^2y(x)}{dx^2} = 0$:

Integrate both sides with respect to x:

$\frac{dy(x)}{dx} = \int 0\,dx = c_1$, where c_1 is an arbitrary constant.

Substitute $\frac{dy(x)}{dx} = c_1$ into $y(x) = x\,\frac{dy(x)}{dx} - e^{\frac{dy(x)}{dx}}$:

$y(x) = -e^{c_1} + c_1\,x$

For $x - e^{\frac{dy(x)}{dx}} = 0$:

Solve for $\frac{dy(x)}{dx}$:

$$\frac{d\,y(x)}{d\,x} = \log(x)$$

Substitute into $y(x) = x\,\frac{dy(x)}{dx} - e^{\frac{dy(x)}{dx}}$:

$y(x) = -x + x\log(x)$

Collect solutions:

Answer:

$$y(x) = -e^{c_1} + c_1\,x \text{ or } y(x) = -x + x\log(x)$$