Answer on solution 38331 – Math - Calculus

Minimize the function f subject to two constraints: $f(x, y, z) = xyz$ on the intersection of

$x^2 + y^2 - 1 = 0$ and $x - z = 0$.

Solution:

The Lagrange function is

$$L = xyz + \lambda(x^2 + y^2 - 1) + \mu(x - z).$$

Thus we have the following system of equations

$$\begin{cases} \frac{\partial L}{\partial x} = yz + 2\lambda x + \mu = 0, \\ \frac{\partial L}{\partial y} = xz + 2\lambda y = 0, \\ \frac{\partial L}{\partial z} = xy - \mu = 0, \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0, \\ \frac{\partial L}{\partial \mu} = x - z = 0, \end{cases} \Rightarrow \begin{cases} yz + 2\lambda x + \mu = 0, \\ xz + 2\lambda y = 0, \\ xy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow \begin{cases} \mu + 2\lambda x + \mu = 0, \\ xz + 2\lambda y = 0, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases}$$

From the first equation

$$\mu + 2\lambda x + \mu = 0,$$

$$2\mu + 2\lambda x = 0,$$

$$2xy + 2\lambda x = 0,$$

$$x(y + \lambda) = 0,$$

$$x = 0 \quad or \quad y + \lambda = 0.$$

If $x = 0$ then we obtain

$$z = x = 0, \mu = xy = 0,$$

and

$$y^2 = 1 - x^2,$$

$$y^2 = 1,$$

$$y = \pm 1.$$

Furthermore

$$xz + 2\lambda y = 0,$$

$$\lambda y = 0.$$

Since $y \neq 0$ then

$$\lambda = 0.$$

So $X_1 = (0; 1; 0)$, $X_2 = (0; -1; 0)$ and

$$f(X_1) = f(X_2) = 0.$$

If $y + \lambda = 0$ then we have

$$\begin{cases} xz + 2\lambda y = 0, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow \begin{cases} xz - 2y^2 = 0, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow \begin{cases} x^2 - 2(1 - x^2) = 0, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow \begin{cases} x^2 - 2 + 2x^2 = 0, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3x^2 = 2, \\ zy = \mu, \\ y^2 = 1 - x^2, \\ x = z, \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{\frac{2}{3}}, \\ \mu = zy, \\ y^2 = \frac{1}{3}, \\ z = \pm\sqrt{\frac{2}{3}}, \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{\frac{2}{3}}, \\ \mu = \pm\frac{\sqrt{2}}{3}, \\ y = \pm\sqrt{\frac{1}{3}}, \\ z = \pm\sqrt{\frac{2}{3}}. \end{cases}$$

Since we have to minimize the function $f(x, y, x)$ then

$$X_{min}^{(1)} = \left(-\sqrt{\frac{2}{3}}; -\sqrt{\frac{1}{3}}; -\sqrt{\frac{2}{3}}\right)$$

$$X_{min}^{(2)} = \left(\sqrt{\frac{2}{3}}; -\sqrt{\frac{1}{3}}; \sqrt{\frac{2}{3}}\right)$$

and

$$f_{min} = f\left(X_{min}^{(1)}\right) = f\left(X_{min}^{(2)}\right) = -\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{3}} \cdot \sqrt{\frac{2}{3}} = -\frac{2}{3\sqrt{3}}.$$

Answer:

$$f_{min} = -\frac{2}{3\sqrt{3}}$$

$$X_{min}^{(1)} = \left(-\sqrt{\frac{2}{3}}; -\sqrt{\frac{1}{3}}; -\sqrt{\frac{2}{3}}\right), X_{min}^{(2)} = \left(\sqrt{\frac{2}{3}}; -\sqrt{\frac{1}{3}}; \sqrt{\frac{2}{3}}\right)$$