Show that the set

$$W = \{(x_1, x_2, x_3, x_4): x_1 + x_2 + x_3 + x_4 = 0\}$$

is a subspace of $\mathbb{R}^4$. Verify that $(1, — 1, 1, — 1)$ and $(1, 0, 0, — 1)$ are in W. Find a basis of W containing $(1, — 1, 1, — 1)$ and $(1, 0, 0, — 1)$.

Solution.

We need to prove that $\forall x, y \in W, \forall \alpha, \beta \in \mathbb{R}: \alpha x + \beta y \in W$.

$$x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \Rightarrow$$

$$\Rightarrow \alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4);$$

Assume that $x, y \in W$. Hence:

$$\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2 + \alpha x_3 + \beta y_3 + \alpha x_4 + \beta y_4 =$$

$$= \alpha(x_1 + x_2 + x_3 + x_4) + \beta(y_1 + y_2 + y_3 + y_4) = \alpha \cdot 0 + \beta \cdot 0 = 0 \Rightarrow \alpha x + \beta y \in W.$$

$$(x_1, x_2, x_3, x_4) = (1, -1, 1, -1) \Rightarrow x_1 + x_2 + x_3 + x_4 = 1 - 1 + 1 - 1 = 0 \Rightarrow$$

$$\Rightarrow (1, -1, 1, -1) \in W.$$

$$(x_1, x_2, x_3, x_4) = (1, 0, 0, -1) \Rightarrow x_1 + x_2 + x_3 + x_4 = 1 + 0 + 0 - 1 = 0 \Rightarrow$$

$$\Rightarrow (1, 0, 0, -1) \in W.$$

Consider the following vectors:

$$e_1 = (1, -1, 1, -1), e_2 = (1, 0, 0, -1), e_3 = (0, 1, 0, -1);$$

Prove that $\{e_1, e_2, e_3\}$ is a basis in W. We need to prove that $\{e_1, e_2, e_3\}$ is linearly independent and $\forall x \in W: \exists\ a_1, a_2, a_3 \in \mathbb{R}: x = a_1 e_1 + a_2 e_2 + a_3 e_3$.

$$a_1 e_1 + a_2 e_2 + a_3 e_3 = (0,0,0) \Rightarrow (a_1 + a_2, -a_1 + a_3, a_1, -a_1 - a_2 - a_3) = (0,0,0,0) \Rightarrow$$

$$\Rightarrow \begin{cases} a_1 + a_2 = 0 \\ -a_1 + a_3 = 0 \\ a_1 = 0 \\ -a_1 - a_2 - a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = 0 \\ a_3 = 0 \\ a_1 = 0 \end{cases} \Rightarrow \{e_1, e_2, e_3\} \text{ is linearly independent.}$$

$$x \in W \Rightarrow x = (x_1, x_2, x_3, -x_1 - x_2 - x_3) = x_1(1,0,0,-1) + x_2(0,1,0,-1) + x_3(0,0,1,-1) =$$

$$= x_1 e_2 + x_2 e_3 + x_3(e_1 - e_2 + e_3) = x_3 e_1 + (x_1 - x_3) e_2 + (x_2 + x_3) e_3.$$

Hence, $\{e_1, e_2, e_3\}$ is a basis in W.