Prove the following identities

a) $\sin(x+y)\sin(x-y) = sin^2x - sin^2 y$
b) $\sin(2x) + \sin(4x) + \sin(6x) = 4\cos x \cos(2x)\sin(3x)$

Solution:

a) We'll use next identities

$$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \sin\beta \cdot \cos\alpha,$$

$$cos^2\alpha = 1 - sin^2\alpha.$$

Thus we have

$$\sin(x+y)\sin(x-y) = (\sin x \cdot \cos y + \sin y \cdot \cos x) \cdot (\sin x \cdot \cos y - \sin y \cdot \cos x) =$$

$$= \sin^2 x \cdot \cos^2 y - \sin^2 y \cdot \cos^2 x = \sin^2 x \cdot (1 - \sin^2 y) - \sin^2 y \cdot (1 - \sin^2 x) =$$

$$= \sin^2 x - \sin^2 x \cdot \sin^2 y - \sin^2 y + \sin^2 y \cdot \sin^2 x = \sin^2 x - \sin^2 y.$$

b) We'll use next identities

$$\sin(2\alpha) = 2\sin\alpha \cdot \cos\alpha,$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}.$$

Thus we have

$$\sin(2x) + \sin(4x) + \sin(6x) = (\sin(2x) + \sin(6x)) + \sin(4x) =$$

$$= 2\sin\frac{2x+6x}{2} \cdot \cos\frac{2x-6x}{2} + \sin(4x) = 2\sin(4x) \cdot \cos(2x) + \sin(4x) =$$

$$= 2\sin(4x) \cdot \cos(2x) + 2\sin(2x) \cdot \cos(2x) = 2\cos(2x)\,(\sin(4x) + \sin(2x)) =$$

$$= 2\cos(2x) \cdot 2\sin\frac{2x+4x}{2} \cdot \cos\frac{2x-4x}{2} = 4\cos(2x) \cdot \sin(3x) \cdot \cos x =$$

$$= 4\cos x \cos(2x)\sin(3x).$$