Solve the following Cauchy problem for the nonhomogeneous wave equation.

$$u_{tt}(x,t) - u_{xx}(x,t) = 1, -\infty < x < +\infty, t > 0,$$
$$u(x,0) = x^2,$$
$$u_t(x,0) = 1.$$

Solution:

For Cauchy problem

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) = f(x,t), -\infty < x < +\infty, t > 0,$$
$$u(x,0) = \varphi(x),$$
$$u_t(x,0) = \psi(x)$$

we have the next solution

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)dz + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\tau)dz\,d\tau.$$

In our case

$$a = 1,$$

$$f(x,t) = 1,$$

$$\varphi(x) = x^2,$$

$$\psi(x) = 1.$$

Thus

1) $\frac{\varphi(x+at)+\varphi(x-at)}{2} = \frac{(x+t)^2+(x-t)^2}{2} = x^2 + t^2$

2) $\frac{1}{2a}\int_{x-at}^{x+at} \psi(z)dz = \frac{1}{2}\int_{x-t}^{x+t} dx = \frac{1}{2}\big(x + t - (x-t)\big) = t$

3) $\frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\tau)dz\,d\tau = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} dz\,d\tau = \frac{1}{2}\int_0^t (x + t - \tau - x + t - \tau)d\tau =$

$$= \int_0^t (t-\tau)d\tau = (t\tau - \frac{\tau^2}{2})\Big|_0^t = \frac{t^2}{2}$$

Thus we have

$$u(x,t) = x^2 + t^2 + t + \frac{t^2}{2} = x^2 + \frac{3}{2}t^2 + t.$$

Answer:

$$u(x,t) = x^2 + \frac{3}{2}t^2 + t.$$