Question 1. *Find the subring of the ring* $\mathbb{Z} \times \mathbb{Z}$ *that is not an ideal of* $\mathbb{Z} \times \mathbb{Z}$.

Solution. Consider the subset

$$D = \{(n, n) \mid n \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}.$$

It is a subring of $\mathbb{Z} \times \mathbb{Z}$. Indeed, it is a subgroup of additive group of $\mathbb{Z} \times \mathbb{Z}$, because

$$(n, n) - (m, m) = (n - m, n - m) \in D$$

for arbitrary $(n, n), (m, m) \in I$. Furthermore, D is closed under multiplication:

$$(n, n) \cdot (m, m) = (nm, nm) \in D$$

for all $(n, n), (m, m) \in I$. Finally, D contains the identity $(1, 1)$ of $\mathbb{Z} \times \mathbb{Z}$. The last fact also shows that D cannot be an ideal of $\mathbb{Z} \times \mathbb{Z}$ because $D \neq \mathbb{Z} \times \mathbb{Z}$. This can be demonstrated by the following observation:

$$(1, 1) \cdot (1, 0) = (1, 0) \notin D,$$

while $(1, 1) \in D$.
Answer: for example, $D = \{(n, n) \mid n \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$. □