Sample: Matrix Tensor Analysis - Statements with Matrices

- *Let* $A = \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 2 & 1 \\ \alpha & \alpha & 1 \end{bmatrix}$.

a. For which numbers α will A be singular?

b. For all numbers α not on your list in part a, we can solve $A\boldsymbol{x} = \boldsymbol{b}$ for every vector $b \in R^3$. For each of the numbers α on your list, give the vectors $\boldsymbol{b}$ for which we can solve $A\boldsymbol{x} = \boldsymbol{b}$.

Solution.

a. A matrix is singular if and only if its determinant is 0. Thus, find values of α for that $\det(A) = 0$.

$\det(A) = \begin{vmatrix} 1 & \alpha & \alpha \\ \alpha & 2 & 1 \\ \alpha & \alpha & 1 \end{vmatrix} =$

$= 1 * 2 * 1 + \alpha * \alpha * 1 + \alpha * \alpha * \alpha - \alpha * 2 * \alpha - 1 * 1 * \alpha - \alpha * \alpha * 1 =$

$= 2 + \alpha^2 + \alpha^3 - 2\alpha^2 - \alpha - \alpha^2 =$

$= \alpha^3 - 2\alpha^2 - \alpha + 2 = 0$

Solve the equation above:

$$\alpha^3 - 2\alpha^2 - \alpha + 2 = 0$$
$$\alpha^2(\alpha - 2) - (\alpha - 2) = 0$$
$$(\alpha - 1)(\alpha + 1)(\alpha - 2) = 0$$
$$\alpha = -1, 1, 2$$

b. For $\alpha = -1$ we get:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Form the augmented matrix [A | b] and put it in echelon form:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & b1 \\ -1 & 2 & 1 & b2 \\ -1 & -1 & 1 & b3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & b1 \\ 0 & 1 & 0 & b2 + b1 \\ 0 & -2 & 0 & b3 + b1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & b1 \\ 0 & 1 & 0 & b2 + b1 \\ 0 & 0 & 0 & b3 + b1 + 2(b2 + b1) \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & -1 & -1 & b1 \\ 0 & 1 & 0 & b2 + b1 \\ 0 & 0 & 0 & b3 + 2b2 + 3b1 \end{array}\right)$$

We can see the system has a solution if and only if

$$b3 + 2b2 + 3b1 = 0$$

Or:

$$b3 = -2b2 - 3b1$$

Thus:

$$b = \begin{pmatrix} b1 \\ b2 \\ -2b2 - 3b1 \end{pmatrix}$$

The corresponding normal vector is:

$$b = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

For $\alpha = 1$ we get:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Form the augmented matrix [A | b] and put it in echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b1 \\ 1 & 2 & 1 & b2 \\ 1 & 1 & 1 & b3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & b1 \\ 0 & 1 & 0 & b2-b1 \\ 0 & 0 & 0 & b3-b1 \end{array}\right)$$

We can see the system has a solution if and only if

$$b3 - b1 = 0$$

Or:

$$b3 = b1$$

Thus:

$$b = \begin{pmatrix} b1 \\ b2 \\ b1 \end{pmatrix}$$

The corresponding normal vector is:

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\alpha = 2$ we get:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

Form the augmented matrix [A | b] and put it in echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & b1 \\ 2 & 2 & 1 & b2 \\ 2 & 2 & 1 & b3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & b1 \\ 2 & 2 & 1 & b2 \\ 0 & 0 & 0 & b3-b2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & b1 \\ 0 & -2 & -3 & b2-2b1 \\ 0 & 0 & 0 & b3-b2 \end{array}\right)$$

We can see the system has a solution if and only if

$$b3 - b2 = 0$$

Or:

$$b3 = b2$$

Thus:

$$b = \begin{pmatrix} b1 \\ b2 \\ b2 \end{pmatrix}$$

The corresponding normal vector is:

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- *Suppose A is an $m \times n$ matrix with rank m and $v_1, \dots, v_k \in R^n$ are vectors with Span $(v_1, \dots, v_k) = R^n$. Prove that Span $(Av_1, \dots, Av_k) = R^m$.*

Solution.

$Rank(A) = m$, thus A has m linearly independent columns. Also we know that $Span(v_1, v_2, \dots, v_k) = R^n$. Thus, a set $(Av_1, Av_2, \dots, Av_k)$ contains m linearly independent vectors.

Each of the products Av_i is a vector of size m (according to rules of matrix multiplication).

Thus, the set $(Av_1, Av_2, \dots, Av_k)$ contains m linearly independent vectors of size m. And it means that $Span(Av_1, Av_2, \dots, Av_k) = R^m$.

- *Let A be an $m \times n$ matrix with column vectors $a_1, \dots, a_n \in R^m$.*
  - *a. Suppose $a_1 + \dots + a_n = 0$. Prove that rank(A)$< n$.*
  - *b. More generally, suppose there is some linear combination $c_1a_1 + \dots + c_na_n = 0$, where some $c_i \neq 0$. Prove that rank(A)$< n$.*

Solution.

a. $a_1 + \dots + a_n = 0 => a_n = -a_1 - a_2 - \dots - a_{n+1}$

We can see that the last column (at least one column) can be represented as a linear combination of other columns. Thus, number of linearly independent columns is less than n. That's why $rank(A) < n$.

b. Similarly to part (a) we can see that:

$$a_n = -\frac{c_1}{c_n}a_1 - \frac{c_2}{c_n}a_2 - \dots - \frac{c_{n-1}}{c_n}a_{n-1}$$

Now let

$$c_1' = -\frac{c_1}{c_n}, \dots, c_{n-1}' = -\frac{c_{n-1}}{c_n}$$

We can see that the last column (at least one column) is represented as a linear combination of other columns:

$$a_n = c_1'a_1 + \dots + c_{n-1}'a_{n-1}$$

Thus, number of linearly independent columns is less than n. That's why $rank(A) < n$.

- *Prove or give counterexample. Assume all the matrices are $n \times n$.*
  a. If $AB = CB$ and $B \neq 0$, then $A = C$.
  b. If $A^2 = A$ then $A \neq 0$ or $A = I$.
  c. $(A + B)(A - B) = A^2 - B^2$.
  d. If $AB = CB$ and B is nonsingular, then $A = C$.
  e. If $AB = BC$ and B is nonsingular, then $A = C$.

Solution.

a. $AB = CB => AB - CB = 0 => (A - C)B = 0.$

$$B \neq 0 => A - C = 0 => A = C$$

b. The counterexample is:

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

c. The counterexample is:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

d. $AB = CB => AB - CB = 0 => (A - C)B = 0$

$$B\ is\ nonsingular => B \neq 0 => A - C = 0 => A = C$$

e. The counterexample is:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- *Find all* 2×2 *matrices* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *satisfying*

  a. $A^2 = I_2$

  b. $A^2 = 0$

  c. $A^2 = -I_2$

Solution.

a. $A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We get the system of equations:

$$\begin{cases} a^2 + bc = 1 \\ ab + bd = 0 \\ ac + cd = 0 \\ bc + d^2 = 1 \end{cases}$$

It is clear that $A = I_2$ is a solution of equation $A^2 = I_2$. Now, look for other solutions. Assume $b, c \neq 0$. Thus:

$$\begin{cases} a^2 + bc = 1 \\ ab + bd = 0 \\ ac + cd = 0 \\ bc + d^2 = 1 \end{cases} => \begin{cases} bc = 1 - a^2 \\ a + d = 0 \\ a + d = 0 \\ bc = 1 - d^2 \end{cases} => \begin{cases} a = -d \\ b = \dfrac{1 - d^2}{c} \end{cases}$$

So, $A = I_2$ and each matrix

$$A = \begin{pmatrix} -d & \dfrac{1 - d^2}{c} \\ c & d \end{pmatrix}$$

for every d and $c \neq 0$ are solutions of the equation $A^2 = I_2$.

b. Similarly to the part (a) we get a system of equations:

$$\begin{cases} a^2 + bc = 0 \\ ab + bd = 0 \\ ac + cd = 0 \\ bc + d^2 = 0 \end{cases}$$

For b = 0 we get:

$$\begin{cases} a^2 = 0 \\ 0 = 0 \\ c * 0 = 0 \\ d^2 = 0 \end{cases} => a, d = 0, c\ can\ be\ any$$

So, $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ is a solution.

For b = 0 we get:

$$\begin{cases} a^2 = 0 \\ b * 0 = 0 \\ 0 = 0 \\ d^2 = 0 \end{cases} => a, d = 0, b\ can\ be\ any$$

So, $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a solution too.

For $b, c \neq 0$:

$$\begin{cases} a^2 + bc = 0 \\ ab + bd = 0 \\ ac + cd = 0 \\ bc + d^2 = 0 \end{cases} => \begin{cases} bc = -a^2 \\ a + d = 0 \\ a + d = 0 \\ bc = -d^2 \end{cases} => \begin{cases} a = -d \\ b = \dfrac{-d^2}{c} \end{cases}$$

So, $A = \begin{pmatrix} -d & \frac{-d^2}{c} \\ c & d \end{pmatrix}$ is also a solution if $c, d \neq 0$.

c. $A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

We get the system of equations:

$$\begin{cases} a^2+bc=-1 \\ ab+bd=0 \\ ac+cd=0 \\ bc+d^2=-1 \end{cases}$$

For b = 0 or c = 0:

$$\begin{cases} a^2=-1 \\ 0=0 \\ ac+cd=0 \\ d^2=-1 \end{cases}$$

It is clear that there is no solution.

Assume $b, c \neq 0$. Thus:

$$\begin{cases} a^2+bc=-1 \\ ab+bd=0 \\ ac+cd=0 \\ bc+d^2=-1 \end{cases} => \begin{cases} bc=-1-a^2 \\ a+d=0 \\ a+d=0 \\ bc=-1-d^2 \end{cases} => \begin{cases} a=-d \\ b=\dfrac{-1-d^2}{c} \end{cases}$$

So each matrix

$$A = \begin{pmatrix} -d & \dfrac{-1-d^2}{c} \\ c & d \end{pmatrix}$$

for every d and $c \neq 0$ are solutions of the equation $A^2 = I_2$.