Sample: Field Theory - Particle Physics

N2 C and D

Here I will give explanation for general case of fermion field transforming as vector under some arbitrary Lie group action. For any Lie group one can introduce its generators (its Lie algebra) - an infinitesimal transformation along one of group parameters. For example, for SU(2) group, its generators will be Pauli matrices. We will denote the elements of the generating Lie algebra elements as T_a in general case. The basis of generators have one very nice property, commutation of the two elements of algebra also is element of algebra:

$$[T^a, T^b] = if^{abc}T^c$$

Here f^{abc} are numbers, called algebra structure constants.

Now, the question is, why do we need the covariant derivative at all? Our goal, when introducing this structure is to make it in such way, that expression $\bar{\psi}D_\mu\psi$ is kept constant under the transformation of gauge group. For example, if a gauge transformation is given by

$$\psi \mapsto e^{i\Lambda}\psi$$

and for the gauge potential

$$A_\mu \mapsto A_\mu + \frac{1}{e}(\partial_\mu\Lambda)$$

then D_μ transforms as

$$D_\mu \mapsto \partial_\mu - ieA_\mu - i(\partial_\mu\Lambda)$$

and $D_\mu\psi := \partial_\mu -$ transforms as

$$D_\mu\psi \mapsto e^{i\Lambda}D_\mu\psi$$

and $\bar{\psi} := \psi^\dagger\gamma^0$ transforms as

$$\bar{\psi} \mapsto \bar{\psi}e^{-i\Lambda}$$

so that

$$\bar{\psi}D_\mu\psi \mapsto \bar{\psi}D_\mu\psi$$

and $\bar{\psi}D_\mu\psi$ is therefore gauge invariant in Lagrangian. So, in general case, the special vector fields are introduced, the gauge vector fields A_μ^{ij}. Here i and j and indexes of the Lie algebra and μ is coordinate index. So, in general case the form of the covariant derivative is

$$D_\mu^{ij} = I\partial_\mu - igT^{ij}A_\mu^{ij}$$

where I is unity operator, and g is gauge coupling. The field strength tensor $F_{\mu\nu}$ in generals case are defined as

$$F_{\mu\nu}^{ij} = \frac{1}{ig}[D_\mu^{ij}D_\nu^{ij}] = \partial_\mu A_\nu^{ij} - \partial_\nu A_\mu^{ij} + gf^{ijcd}A_\mu^{cd}A_\nu^{cd}$$

So all we have to do is to find generators of Lie algebra of SO(n) and their structure constant. But you already have them in your problemsheet, the generators are matrices

$$(T^{ij})_{ab} = -i(\delta_a^i\delta_b^j - \delta_b^i\delta_a^j)$$

And so, for $D_\mu^{ab}\psi_b$ we have

$$D_\mu^{ab}\psi_b = ((D_\mu\psi)_a)_{sm} = \delta_s^m\partial_\mu\psi_a + g(\psi_m\delta_s^a + \psi_s\delta_m^a)$$

For the field strength tensor we have

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + g(\delta_{ac}T_{bd} + \delta_{bd}T_{ac} - \delta_{ad}T_{bc} - \delta_{bc}T_{ad})A_\mu^{cd}A_\nu^{cd} =$$

$$= \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + g(A^{ad}A^{ad}T_{bd} + A^{cb}A^{cb}T_{ac} - A^{ac}A^{ac}T_{bc} - A^{bd}A^{bd}T_{ad})$$

N3

The field equations for A_μ have the form

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\mathcal{L}}{\partial_\nu A_\mu}\right) = 0$$

And so, as we do the evaluation we see, indeed

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = \frac{\partial(\bar{\psi}(i\gamma^\mu\partial_\mu - \gamma^\mu igA_\mu^a t^a - m)\psi)}{\partial A^\mu} = -ig\bar{\psi}\gamma^\mu t^a\psi$$

$$\partial_\nu\left(\frac{\mathcal{L}}{\partial_\nu A_\mu}\right) = \partial_\nu\frac{1/4\partial(F_{\mu\nu}^a)^2}{\partial_\nu A_\mu} = |F_{\nu\mu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c| =$$

$$= \partial_\nu F^a_{\nu\mu} + g f^{abc} A^{\nu b} F^c_{\mu\nu} = (D^\mu F_{\nu\mu})^a$$

And hence we get field equation we need

$$(D^\mu F_{\nu\mu})^a = -g j^a_\nu, \qquad j^a_\nu = \bar{\psi}\gamma^\mu t^a \psi$$

N4

A

We will suppose, that A^a_μ are gauge fields that correspond to weak isospin T^3 (SU(2) symmetry) and B_μ corresponds to the weak hypercharge Y (U(1) symmetry). If so, then A^a_μ will be present only in those covariant derivatives, that correspondent to fields which have non-zero isospin. B_μ will be present in all covariant derivative. Hence, we will have these derivatives:

for Q_L:

$$D_\mu = \partial_\mu - g t^a A^a_\mu - i g' B_\mu = \partial_\mu + g \frac{\sigma^a}{2} A^a_\mu - i g' Y^L B_\mu$$

Here we used the fact that sigma-matrices are generators for SU(2) and generator for U(1) is just complex number i. In all other covariant derivatives there will be only B_μ:

$$(D_\mu)^k = \partial_\mu - i g' Y^k B_\mu$$

here k denotes index that corresponds to particular field, and Y^k is correspondent hypercharge of that field. (L, 3L, 1R, 2R, 3R)

B and C

Firstly let us recall what is Higgs mechanism in Standard Model and how it works. Mathematically, it nothing but introducing new field variables instead of old ones. Indeed, in Standard model we have

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_Y + \mathcal{L}_\phi + \mathcal{L}_V$$

Here $\mathcal{L}_\phi$ is Higgs sector, $\mathcal{L}_f$ is kinetic part of fermion sector, $\mathcal{L}_V$ is gauge fields sector and $\mathcal{L}_Y$ is interaction sector. So the mechanism is to transform these parts

$$\mathcal{L}_f = -\sum_{m=1}^{3} [\bar{Q}_m \gamma^\mu D_m u Q_m + \bar{L}_m \gamma^\mu D_m u L_m + \bar{U}_m \gamma^\mu D_m u U_m + \bar{D}_m \gamma^\mu D_m u \mathcal{D}_m + \bar{E}_m \gamma^\mu D_m u E_m]$$

$$\mathcal{L}_Y = -\sqrt{2} \sum_{m,n=1}^{3} [Y^u_{mn}(i\tau_2 \phi^\dagger)\bar{Q}_m U_n + Y^d_{mn}\phi \bar{Q}_m \mathcal{D}_n + Y^e_{mn}\phi \bar{L}_m E_n]$$

into a next form

$$\mathcal{L}_f+\mathcal{L}_Y=\sum_r \bar{\psi}_r[i\gamma^\mu\partial_\mu-m_r(1+\frac{H}{v})]\psi_r-\frac{g}{\sqrt{2}}[J_W^{\mu\dagger}W_\mu^+ +J_W^\mu W_\mu^- +J_A^\mu A_\mu+J_Z^\mu Z_\mu]$$

Here H is higgs field, v is its vacuum expectation value, r is summation over fermions, $J^\pm$ is charged currents, J_A is axial current and J_Z is neutral current. We will give here expression only for that last one

$$J_Z=\sum_f \bar{\psi}_f\gamma^\mu[g_L^f P_L+g_R^f P_R]\psi_f=\sum_f \bar{\psi}_f\gamma^\mu\frac{g_V^f-g_A^f\gamma^5}{2}\psi_f$$

where f is summation over fermions and g_V and g_A for every fermion are defined as:

$$g_f^V=\sqrt{2}\frac{T_f^3-2\sin^2\theta_w Q_f}{\cos\theta_W},\qquad g_f^A=\sqrt{2}\frac{T_f^3}{\cos\theta_W}$$

$$\cos\theta=\frac{g}{\sqrt{g^2+g'^2}},\ \sin\theta_W=\frac{g'}{\sqrt{g^2+g'^2}},\ Q^f=T_3+Y/2$$

What is the difference between your model and the Standard model? In your model there are only three quarks and no other fermions and in your model the T^3-charges of quarks are different - in real world all of quarks do have T^3-charge, while here its just the L-doubled of 1 and 2 quark. Hence, all we have to do is to insert into J_Z our values of g^V and g^A, the form of J_Z will be the same. So here are those values

$$g_{1L}^V=\sqrt{2}\frac{1/2-2\sin^2\theta_W(1/2+1/6)}{\cos\theta_W},\quad g_{1L}^A=\frac{1/2}{\cos\theta_W}$$

$$g_{2L}^V=\sqrt{2}\frac{-1/2-2\sin^2\theta_W(-1/2+1/6)}{\cos\theta_W},\quad g_{2L}^A=\frac{-1/2}{\cos\theta_W}$$

$$g_{3L}^V=\sqrt{2}\frac{0-2\sin^2\theta_W(0-1/3)}{\cos\theta_W},\quad g_{3L}^A=0$$

$$g_{1R}^V=\sqrt{2}\frac{0-2\sin^2\theta_W(0+2/3)}{\cos\theta_W},\quad g_{1R}^A=0$$

$$g_{2R}^V=\sqrt{2}\frac{0-2\sin^2\theta_W(0-1/3)}{\cos\theta_W},\quad g_{2R}^A=0$$

$$g_{2R}^{V} = \sqrt{2}\frac{0 - 2\sin^2\theta_W(0 - 1/3)}{\cos\theta_W}, \quad g_{3R}^{A} = 0$$

And finally. Because of these T^3 charges of quarks in Standard Model, there forbidden transformations for them - "up" like quarks can not transform to "down" type. In your model, quarks of type 3L, 1R, 2R, 3R have the same T^3 charge, zero, which means they can transform each to another. If we take into account the mixing, given in part C, then, new interaction that is avaible is transformation "up" quark u to "down" type d and s. The coupling strength will be proportional to mixing cos and sin angles.