Sample: Mat LAB Mathematica MathCAD Maple - Numerical Analysis Using Maple

The embedded Runge-Kutta-Fehlberg method that we will use is given by the Butcher array

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{4} & \frac{1}{4} & & & \\
\frac{27}{40} & -\frac{189}{800} & \frac{729}{800} & & \\
1 & \frac{214}{891} & \frac{1}{33} & \frac{650}{891} & \\
\hline
y_{n+1} & \frac{214}{891} & \frac{1}{33} & \frac{650}{891} & \\
\hat{y}_{n+1} & \frac{533}{2106} & 0 & \frac{800}{1053} & -\frac{1}{78}
\end{array}
\quad . \tag{2.1}
$$

Task 2.1 Write out the formulas for $k_1, \ldots, k_4$, y_{n+1} and $\hat{y}_{n+1}$.

Solution:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right)$$

$$k_3 = f\left(t_n + \frac{27h}{40}, y_n - \frac{189}{800}k_1 + \frac{729}{800}k_2\right)$$

$$k_4 = f\left(t_n + h, y_n + \frac{214}{891}k_1 + \frac{1}{33}k_2 + \frac{650}{891}k_3\right)$$

$$y_{n+1} = y_n + h\left(\frac{214}{891}k_1 + \frac{1}{33}k_2 + \frac{650}{891}k_3\right)$$

$$\widehat{y_{n+1}} = y_n + h\left(\frac{533}{2106}k_1 + \frac{800}{1053}k_3 - \frac{1}{78}k_4\right)$$

Task 2.3 *Write out $R(z)$ in the same way and state the values of the coefficients α_1, α_2 and α_3.*

Task 2.4 *Use Maple to make a plot of the polynomials $R(z)$ and $\hat{R}(z)$ for $z \in [-3, 1]$. Use one graph to show both polynomials together.*

$$k_1 = f(t_n, y_n) = \mu \cdot y_n$$

$$k_2 = f\left(t_n + \frac{h}{4}, y_n + \frac{h}{4}k_1\right) = \mu \cdot \left(y_n + \frac{h}{4}(\mu \cdot y_n)\right)$$

$$k_3 = f\left(t_n + \frac{27h}{40}, y_n - \frac{189}{800}k_1 + \frac{729}{800}k_2\right)$$
$$= \mu \cdot \left(y_n - \frac{189}{800}\mu \cdot y_n + \frac{729}{800}\mu \cdot \left(y_n + \frac{h}{4}(\mu \cdot y_n)\right)\right)$$

$k_4 = f\left(t_n + h, y_n + \frac{214}{891}k_1 + \frac{1}{33}k_2 + \frac{650}{891}k_3\right) = \mu\cdot\left(y_n + \frac{214}{891}\mu\cdot y_n + \frac{1}{33}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right) + \frac{650}{891}\mu\cdot\left(y_n - \frac{189}{800}\mu\cdot y_n + \frac{729}{800}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right)\right)\right)$

$y_{n+1} = y_n + h\left(\frac{214}{891}k_1 + \frac{1}{33}k_2 + \frac{650}{891}k_3\right) = y_n + h\left(\frac{214}{891}\mu\cdot y_n + \frac{1}{33}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right) + \frac{650}{891}\mu\cdot\left(y_n - \frac{189}{800}\mu\cdot y_n + \frac{729}{800}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right)\right)\right) = y_n + \mu\cdot h\cdot y_n + \frac{1}{2}h^2\mu^2 y_n + \frac{117}{704}h^3\mu^3 y_n$

So $y_{n+1} = y_n + \mu\cdot h\cdot y_n + \frac{1}{2}h^2\mu^2 y_n + \frac{117}{704}h^3\mu^3 y_n$

If $z = \mu\cdot h \to y_{n+1} = y_n + \mu\cdot h\cdot y_n + \frac{1}{2}h^2\mu^2 y_n + \frac{117}{704}h^3\mu^3 y_n =$

$= R(z)y_n = y_n\left(1 + z + \frac{1}{2}z^2 + \frac{117}{704}z^3\right)$

So $\alpha_1 = 1, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{117}{704}$

$$\widehat{y_{n+1}} = y_n + h\left(\frac{533}{2106}k_1 + \frac{800}{1053}k_3 - \frac{1}{78}k_4\right) = y_n + h\left(\frac{533}{2106}\mu\cdot y_n + \frac{800}{1053}\mu\cdot\left(y_n - \frac{189}{800}\mu\cdot y_n + \frac{729}{800}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right)\right) - \frac{1}{78}\mu\cdot\left(y_n + \frac{214}{891}\mu\cdot y_n + \frac{1}{33}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right) + \frac{650}{891}\mu\cdot\left(y_n - \frac{189}{800}\mu\cdot y_n + \frac{729}{800}\mu\cdot\left(y_n + \frac{h}{4}(\mu\cdot y_n)\right)\right)\right)\right) = y_n + \mu\cdot h\cdot y_n + \frac{1}{2}h^2\mu^2 y_n + \frac{1}{6}h^3\mu^3 y_n - \frac{3}{1408}h^4\mu^4 y_n$$

$$\widehat{y_{n+1}} = y_n + \mu\cdot h\cdot y_n + \frac{1}{2}h^2\mu^2 y_n + \frac{1}{6}h^3\mu^3 y_n - \frac{3}{1408}h^4\mu^4 y_n$$

If $z = \mu \cdot h \rightarrow \widehat{y_{n+1}} = y_n\left(1 + \mu \cdot h + \frac{1}{2}h^2\mu^2 + \frac{1}{6}h^3\mu^3 - \frac{3}{1408}h^4\mu^4\right) = R(z)y_n =$
$y_n\left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{3}{1408}z^4\right)$

Answer: So $\alpha_1 = 1, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{117}{704}$

Task 2.4 *Use Maple to make a plot of the polynomials $R(z)$ and $\hat{R}(z)$ for $z \in [-3, 1]$. Use one graph to show both polynomials together.*

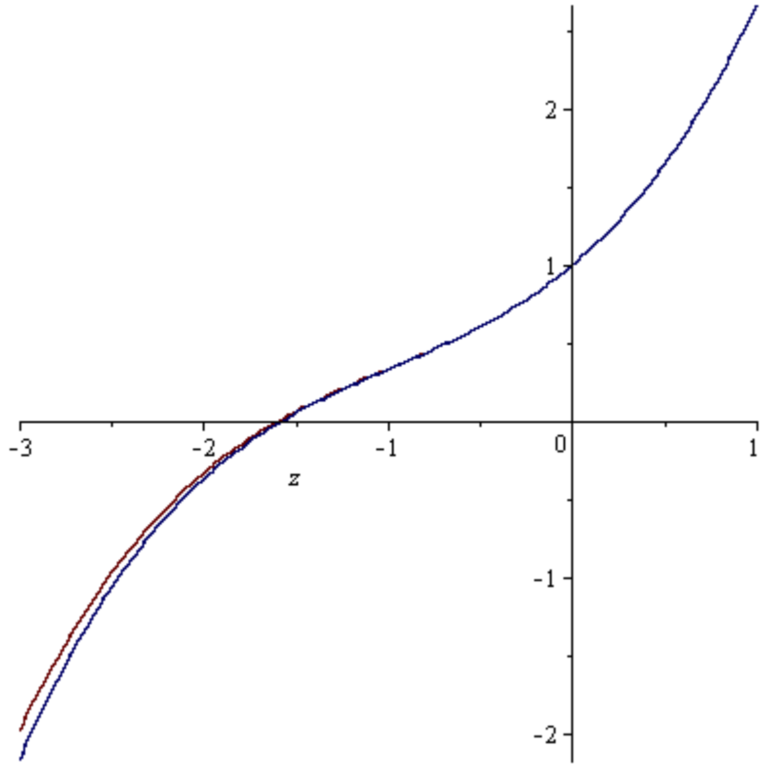
> $R\_Z := 1 + z + \left(\frac{1}{2}\right)\cdot z^2 + \left(\frac{117}{704}\right)\cdot z^3$

$$R\_Z := 1 + z + \frac{1}{2}z^2 + \frac{117}{704}z^3$$

> $R\_Z\_Hat := 1 + z + \left(\frac{1}{2}\right)\cdot z^2 + \left(\frac{1}{6}\right)\cdot z^3 - \left(\frac{3}{1408}\right)\cdot z^4$

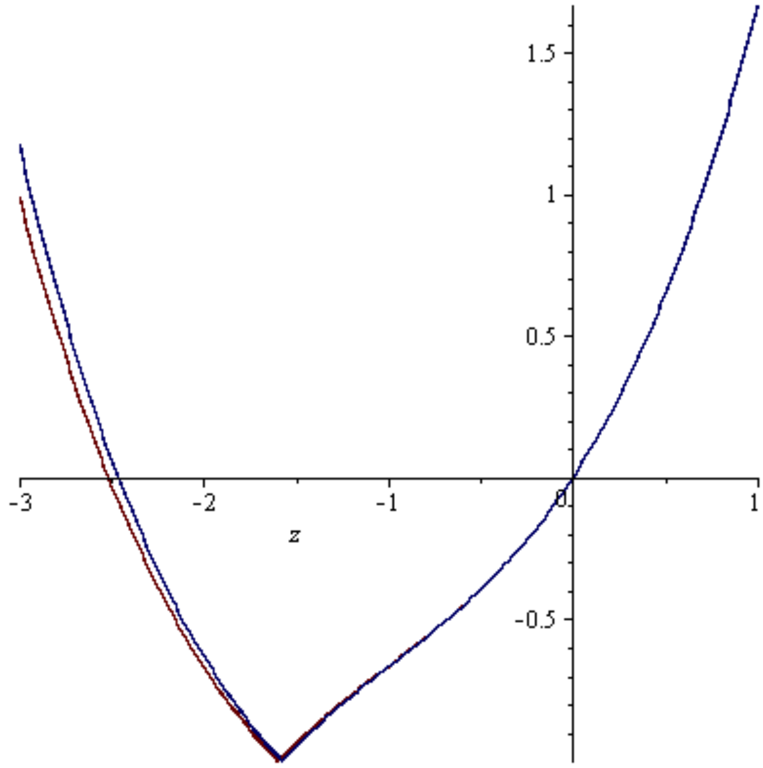
$$R\_Z\_Hat := 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{3}{1408}z^4$$

> plot([R_Z, R_Z_Hat], z = -3 ..1)



Task 2.5 *Use Maple to find the value z^* such that $|R(z)| < 1$ for all values $z \in (z^*, 0)$. In the same way, use Maple to find the value $\hat{z}^*$ such that $|\hat{R}(z)| < 1$ for all values $z \in (\hat{z}^*, 0)$. Hint: You can use the* `fsolve` *command.*

> $plot([\mathrm{abs}(R\_Z) - 1, \mathrm{abs}(R\_Z\_Hat) - 1], z=-3..1)$



> $fsolve(\mathrm{abs}(R\_Z) = 1, z=-3..-1)$

$$-2.517329447$$

> $fsolve(\mathrm{abs}(R\_Z\_Hat) = 1, z=-3..-1)$

$$-2.463900430$$

Task 3.1 *Implement a procedure that takes as arguments the parameters* $a, b, c \in \mathbb{R}$ *and* $z : \mathbb{R} \to \mathbb{R}$*, the initial condition* $t_0, x_0, y_0 \in \mathbb{R}$*, the final time* $t_e \in \mathbb{R}$ *and the number of steps* $N \in \mathbb{N}$ *and calculates an approximation to the solution of the FitzHugh-Nagumo model* (1.1)-(1.2)*. The output of your procedure should be five arrays* $t[0..N], x[0..N], y[0..N], \hat{x}[0..N], \hat{y}[0..N]$ *containing the grid points* t_n *and the numerical approximations* x_n, y_n, $\hat{x}_n$ *and* $\hat{y}_n$*. It may be a good idea to base your implementation on several smaller procedures.*

```
Filling_Arrays := proc(a, b, c, t0, te, x0, y0, N)
local h, x, y, x_hat, y_hat, i, t;
h := evalf((te - t0)/N);
t := Arrays(0..N, i -> t0 + i·h);
x := Arrays(0..N);
y := Arrays(0..N);
x_hat := Arrays(0..N);
y_hat := Arrays(0..N);
x[0] := x0;
x_hat[0] := x0;
y[0] := y0;
y_hat[0] := y0;
for i from 1 to N do
x[i] := 0;
y[i] := 0;
x_hat[i] := 0;
y_hat[i] := 0;
end do;
(t, x, y, x_hat, y_hat);
end proc:
```

>

```
runge_kutta_fehlberg_method := proc(a, b, c, z, t0, te, x0, y0, N)
 local f, g, h, x, y, x_hat, y_hat, i, t, eps, K, G, yt, xhat, yhat, s, xt, fig1, fig2, fig3, fig4, fig5;
f := (t,x,y) → c·( y + x − x^3/3 + z );
 g := (t,x,y) → -(x − a + b·y)/c;
t[0] := t0;
 x[0] := x0;
 y[0] := y0;
 x_hat[0] := x0;
 y_hat[0] := y0;
 h := (te − t0)/N;
eps[abs] := 0.01;
for i from 1 to N do
    K[1] := h·f( t[i−1], x[i − 1], y[i−1] );
    G[1] := h·g( t[i−1], x[i − 1], y[i−1] );

    K[2] := h·f( t[i−1] + (1/4)*h, x[i − 1] + (1/4)*K[1], y[i−1] + (1/4)*G[1] );
    G[2] := h·g( t[i−1] + (1/4)*h, x[i − 1] + (1/4)*K[1], y[i−1] + (1/4)*G[1] );

    K[3] := h·f( t[i−1] + 27/40·h, x[i−1] - 189/800·K[1] + 729/800·K[2], y[i−1] - 189/800·G[1]
    + 729/800·G[2] );
    G[3] := h·g( t[i−1] + 27/40·h, x[i−1] - 189/800·K[1] + 729/800·K[2], y[i−1] - 189/800·G[1]
    + 729/800·G[2] );
     K[4] := h·f( t[i−1] + h, x[i−1] + 214/891·K[1] + 1/33·K[2] + 650/891·K[3], y[i−1]
    + 214/891·G[1] + 1/33·G[2] + 650/891·G[3] );
     G[4] := h·g( t[i−1] + h, x[i−1] + 214/891·K[1] + 1/33·K[2] + 650/891·K[3], y[i−1]
    + 214/891·G[1] + 1/33·G[2] + 650/891·G[3] );
     xt := x[i - 1] + ( 214/891·K[1] + 1/33·K[2] + 650/891·K[3] );
     xhat := x[i - 1] + ( 533/2106·K[1] + 800/1053·K[3] − 1/78·K[4] );

     yt := y[i - 1] + ( 214/891·G[1] + 1/33·G[2] + 650/891·G[3] );
     yhat := y[i - 1] + ( 533/2106·G[1] + 800/1053·G[3] − 1/78·G[4] );

    s := root[4]( (eps[abs]*h) / (2*abs(yt - yhat)) );


    t[i] := t[i - 1] + h;
    x[i] := xt;
    x_hat[i] := xhat;
     y[i] := yt;
     y_hat[i] := yhat;

    if t[i] = te then
         break;
     end if;
 end do:
fig1 := plots[pointplot]( [seq( [t[k], x[k]], k = 1 ..N )], color = red );
 fig2 := plots[pointplot]( [seq( [t[k], y[k]], k = 1 ..N )], color = green );
fig3 := plots[pointplot]( [seq( [t[k], abs(x[k] − x_hat[k])], k = 1 ..N )], color = blue );
fig4 := plots[pointplot]( [seq( [t[k], abs(y[k] − y_hat[k])], k = 1 ..N )], color = magenta );
fig5 := plots[pointplot]( [seq( [x[k], y[k]], k = 1 ..N )] );

 print(fig1)
 print(fig2)
 print(fig3)
 print(fig4)
 print(fig5)
end proc:
```