Sample: Functionla Analysis - Functional Analysis Task

Let $\mathcal{P}$ be a family of separating seminorms on a vector space X. Then to each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, associate a set

$$V(p,n) = \left\{x : p(x) < \frac{1}{n}\right\}$$

Let $\mathcal{B}$ be a collection of finite intersections of $V(p,n)$. Define a set U in X to be open if U is a union of translates of members of $\mathcal{B}$. Then $\mathcal{B}$ is a convex local base for this topology. Prove that this topology makes X into a topological vector space.

Proof. We have to prove that operations addition and of vertors and multiplication by scalars are continuous in that topology.

1) Let $x, y \in X$, $z = x + y$ and U_z be a neighbourhood of z. We have to find neighbourhoods U_x and U_y of x and y such that

$$U_x + U_y \subset U_z,$$

that is

$$x' + y' \in U_z$$

for all $x' \in U_x$ and $y' \in U_y$.

Since we may decrease U_z, it sufffices to consider the case when U_z is a translate of some $V(p,n)$:

$$U_z = V(p,n) + z'$$

for some $z' \in X$.

Moreover, increasing n we can assuem that $z' = z$. Indeed, since $z \in U_z = V(p,n) + z'$, we see that

$$z - z' \in V(p,n),$$

that is

$$p(z - z') < \frac{1}{n}.$$

Take any number $m \in \mathbb{N}$ such that

$$p(z - z') + \frac{1}{m} < \frac{1}{n}.$$

We claim that then

$$z + V(p,m) \subset U_z = V(p,n) + z'.$$

Indeed, if $a \in z + V(p,m)$, so $p(z - a) < \frac{1}{m}$, then

$$p(z' - a) = p(z' - z + z - a) \le p(z' - z) + p(z - a) \le p(z - z') + \frac{1}{m} < \frac{1}{n}.$$

Thus assume that $U_z = z + V(p,n)$ for some p, n.

Put

$$U_x = x + V(p,2n), \qquad U_y = y + V(p,2n).$$

We claim that then

$$U_x + U_y \subset U_z.$$

Indeed, let $x' \in U_x$ and $y' \in U_y$, so

$$p(x - x') < \frac{1}{2n}, \qquad p(y - y') < \frac{1}{2n},$$

Then

$$p(x' + y' - z) = p(x' - x + y' - y + \underbrace{x + y - z}_{=0}) = p(x' - x + y' - y) \le p(x' - x) + p(y' - y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Which means that $x' + y' \in z' + V(p,n) = U_z$.

Thus addition is continuous.

2) Let $x \in X$ and $t \in \mathbb{R}$, U_{tx} be a neighbourhood of tx. We have to find neighbourhoods U_x of x in X and W_t of t in $\mathbb{R}$ such that

$$W_t * U_x \subset U_{tx},$$

that is

$$t'x' \in U_{tx}$$

for all $x' \in U_x$ and $t' \in W_t$.
Again not loosing generality we can assume that

$$U_{tx} = tx + V(p,n).$$

Since p is a seminorm, we have that

$$p(ty) = |t|p(y)$$

for all $y \in X$, whence

$$V(p,n) = tV(p,nt).$$

Indeed, $y \in tV(p,nt)$ if and only if

$$p(y/t) < \frac{1}{nt}$$

which can be rewritten as follows:

$$p(y)/t < \frac{1}{nt},$$

$$p(y) < \frac{1}{n}$$

The latter is equivalent to $y \in V(p,n)$.
In particular,

$$U_{tx} = tx + tV(p,tn) = t(x + V(p,tn)).$$

Thus if we put

$$U_x = x + V(p,tn),$$

then

$$U_{tx} = tU_x.$$

This proves that multiplication by scalars is also continuous and so X is a topological vector space.

Suppose V is an open set containing 0 in a topological vector space X. Prove that if

$$0 < r_1 < r_2 < \cdots$$

and $r_n \to \infty$ as $n \to \infty$, then

$$X = \bigcup_{n=1}^{\infty} r_n V$$

Proof. Let $x \in X$. We have to show that $x \in r_m V$ for some $m \geq 1$.
By assumption V is an open set containing 0. Since $0x = 0$ and the multiplication by scalars in X is continuous there exists $\varepsilon > 0$ such that

$$tx \in V$$

for all $t \in (-\varepsilon, \varepsilon)$.
Since $r_n \to \infty$ increases, there exists $m > 0$ such that

$$0 < \frac{1}{r_m} < \varepsilon$$

Then

$$\frac{1}{r_m} x \in V,$$

whence

$$x \in r_m V \subset \bigcup_{n=1}^{\infty} r_n V,$$

and so $X = \bigcup_{n=1}^{\infty} r_n V$.