Sample: Differential Calculus Equations - Fourier Series

Find Fourier standard coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{0}(-x+1)\,dx + \frac{1}{L}\int_{0}^{L}(-1)\,dx = \frac{1}{L}\left(-\frac{x^2}{2}+x\right)\Big|_{-L}^{0} + \frac{1}{L}(-L) = \frac{1}{L}\left(\frac{L^2}{2}+L-L\right) = \frac{L}{2}$$
$$= 1$$

For positive integer n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi}{L}nx\,dx = \frac{1}{L}\int_{-L}^{0}(-x+1)\cos\frac{\pi}{L}nx\,dx + \frac{1}{L}\int_{0}^{L}(-1)\cos\frac{\pi}{L}nx\,dx$$
$$= (-x+1)\frac{1}{\pi n}\sin\frac{\pi}{L}nx\,\Big|_{-L}^{0} - \int_{-L}^{0}(-1)\frac{1}{\pi n}\sin\frac{\pi}{L}nx\,dx + \frac{1}{L}\int_{0}^{L}(-1)\cos\frac{\pi}{L}nx\,dx$$
$$= \frac{1}{L}(-x+1)\frac{L}{\pi n}\sin\frac{\pi}{L}nx\,\Big|_{-L}^{0} - \frac{1}{L}\left(\frac{L}{\pi n}\right)^2\cos\frac{\pi}{L}nx\,\Big|_{-L}^{0} - \frac{1}{\pi n}\sin\frac{\pi}{L}nx\,\Big|_{0}^{L}$$
$$= -\frac{1}{L}\left(\frac{L}{\pi n}\right)^2(1-(-1)^n) = -\frac{1}{2}\left(\frac{2}{\pi n}\right)^2(1-(-1)^n)$$

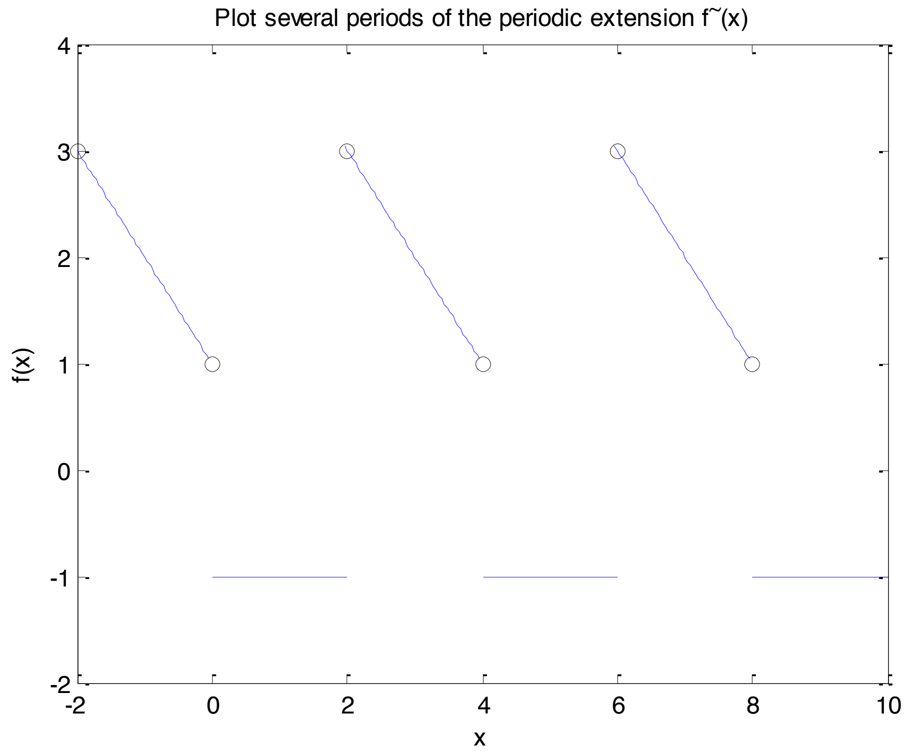
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi}{L}nx\,dx = \frac{1}{L}\int_{-L}^{0}(-x+1)\sin\frac{\pi}{L}nx\,dx + \frac{1}{L}\int_{0}^{L}(-1)\sin\frac{\pi}{L}nx\,dx$$
$$= \frac{1}{L}\left(-(-x+1)\frac{L}{\pi n}\cos\frac{\pi}{L}nx\,\Big|_{-L}^{0} - \int_{-L}^{0}(1)\frac{L}{\pi n}\cos\frac{\pi}{L}nx\,dx + \int_{0}^{L}(-1)\sin\frac{\pi}{L}nx\,dx\right)$$
$$= \frac{1}{L}\left(-(-x+1)\frac{L}{\pi n}\cos\frac{\pi}{L}nx\,\Big|_{-L}^{0} - \left(\frac{L}{\pi n}\right)^2\sin\frac{\pi}{L}nx\,\Big|_{-L}^{0} + \frac{L}{\pi n}\cos\frac{\pi}{L}nx\,\Big|_{0}^{L}\right)$$
$$= \left(\frac{1}{\pi n}\right)((L+2)(-1)^n-2) = \left(\frac{2}{\pi n}\right)(2(-1)^n-1)$$

So the standard Fourier series of function c(x) is (L=2)

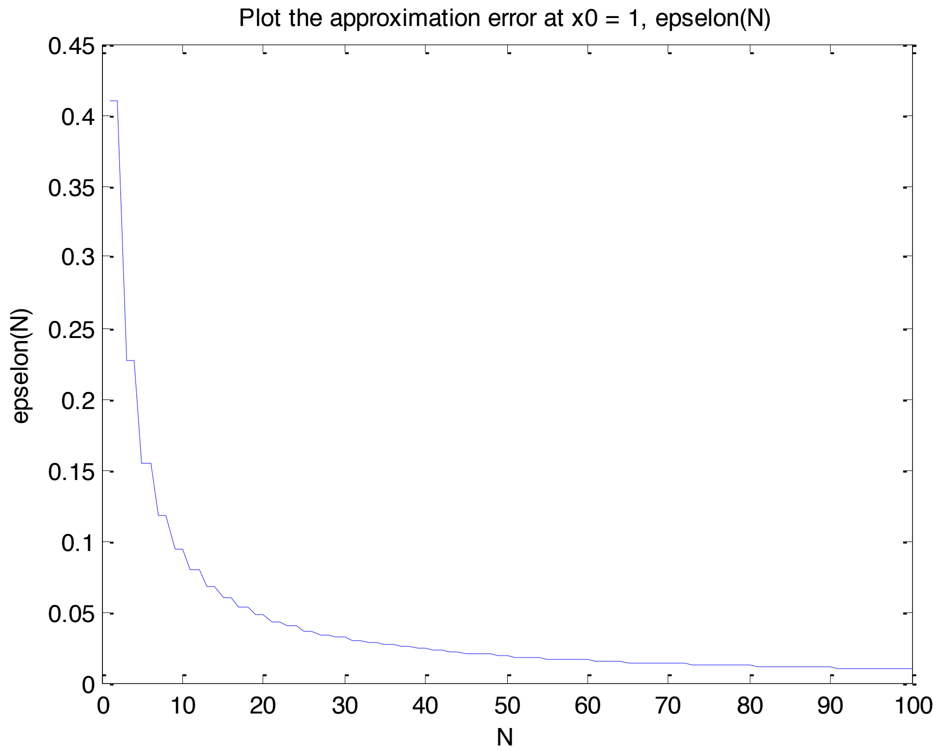
$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{\pi}{L}nx + b_n\sin\frac{\pi}{L}nx$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} -\left(\frac{2}{\pi n}\right)^2(1-(-1)^n)\cos\frac{\pi}{L}nx + \left(\frac{4}{\pi n}\right)(2(-1)^n-1)\sin\frac{\pi}{L}nx$$

2. Plot several periods of the periodic extension ˜ f(x) of your function (or, respectively, odd or even periodic extension, depending on the series you are working with.)

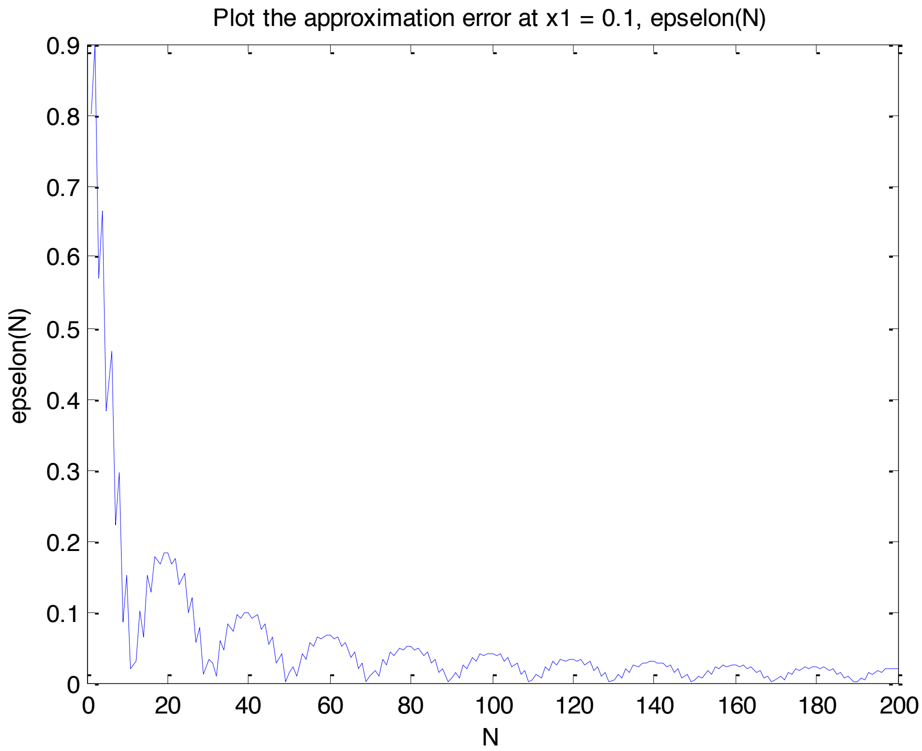
3. Mark points where the periodic extension is discontinuous. Near them, you will observe Gibbs phenomenon, and at such points, the Fourier series will converge to the average of left and right limits of ˜ f(x).



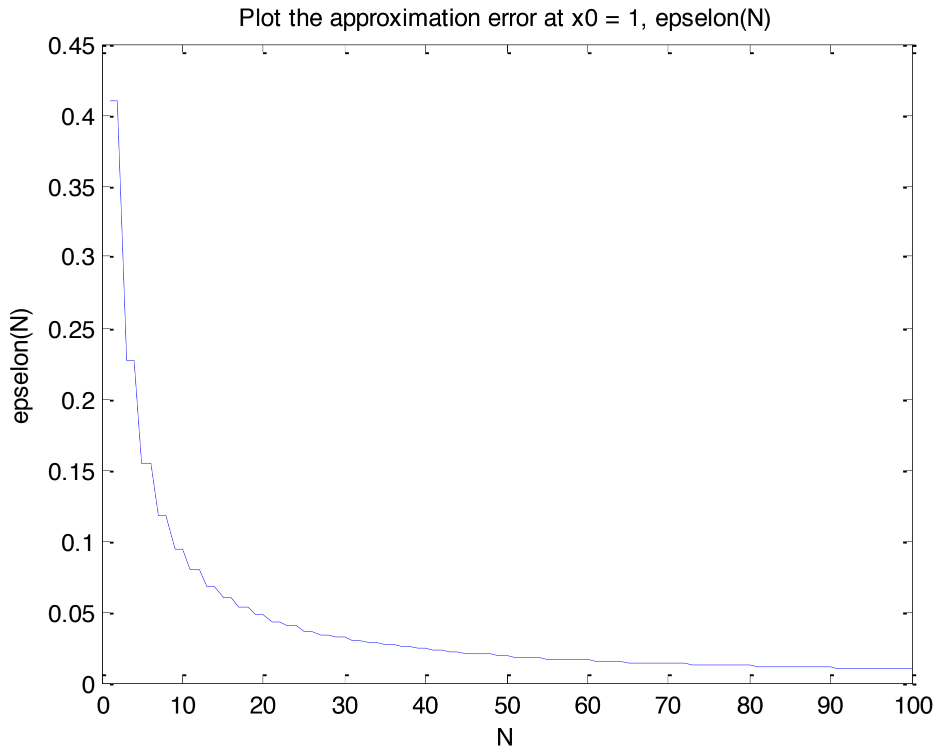
4. Let sN(x) be the partial sum of N terms of the Fourier series. We wish to study the convergence of sN(x) to f(x) at some continuity point x0 (not close to a discontinuity).



5. In the same manner, study the convergence of sN(x) to f(x) at some continuity point x1 close to a discontinuity.



6. In the same manner, study the convergence of sN(x) to the average of values of f(x) at a discontinuity point x.



We find the average of left and right limits of f~(x).

$$\frac{f^{+}(0)+f^{-}(0)}{2} = \frac{1+(-1)}{2} = 0$$

7. Plot the periodic extension f~(x) and several partial sums sN(x) as functions of x.