Sample: Electromagnetism - Electromagnetism Problems

1

When charging the insulated conductor, the charge is distributed only on the surface of the conductor for the following reasons:
1.Because like charges repel, the excessive electrical charges tend to stay as far away from each other as possible, and this corresponds to the uniform charge distribution on the surface;
2.The same can be proven by Gauss's theorem: the field inside the conductor can not exist (otherwise the charges would have moved, and there would be no balance), hence the field flux through any closed surface constructed inside the conductor is zero, Gauss's theorem then implies, that inside the conductor there is no uncompensated electrical charges.

The field is oriented normally, as any other direction would cause free charges to move along the surface.

Now let us find the field near the surface.
We will use Gauss theorem for that. This theorem tells us, that field flux through any close surface is proportional to charge that is inside that surface. Let us choose closed surface so it contains any value S of the conductor's and consist of 6 parts - 4 normal to conductor's surface and 2 parallel (this will be cuboid). The theorem implies:

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

where Q is charged distributed at surface S of conductor. To find field itself we must divide the flux by the surface. So we have

$$E = \frac{\Phi_E}{S} = \frac{Q}{S\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

because Q/S is surface density.

2

To find charge distribution you have to find divergence of the field $divE(r)$. In the spherical coordinates, the divergence will look like

$$\rho = \varepsilon_0 divE = \frac{\varepsilon_0}{r^2}\frac{\partial}{\partial r}[r^2 E(r)] = \frac{\varepsilon_0}{r^2}\frac{\partial}{\partial r}[r^2 E_0 e^{-r/R}] = E_0\varepsilon_0\left(\frac{2e^{-r/R}}{r} + \frac{e^{-r/R}}{R}\right)$$

The potential and field are connected as:

$$E = -grad\phi = - \frac{\partial}{\partial r}\phi$$

Hence:

$$\phi = -\int E(r)dr = E_0 R e^{-r/R}$$

3

We will use superposition principle in this problem. We can consider this system as two whole spheres: one of radius R_1 with charge density ρ and another of radius R_2 with charge density $-\rho$. Because of superposition principle this system will give the same field as the one described in problem. Hence, both the field and its energy will be result of superposition of these two charged spheres.

$$E = E_1 + E_2$$

Using Gauss theorem, we can easily find, that for charged sphere of radius R, field is equal to

$$E(r) = \begin{cases} \frac{\rho}{3\varepsilon_0} r, & r \le R \\ \frac{\rho R^3}{3\varepsilon_0 r^2}, & r > R \end{cases}$$

Now let us find field of the system:

$$E(r) = \begin{cases} 0, & r \le R_2 \\ \frac{\rho}{3\varepsilon_0} r - \frac{\rho R_2^3}{3\varepsilon_0 r^2}, & R_2 < r \le R_1 \\ \frac{\rho R_1^3}{3\varepsilon_0 r^2} - \frac{\rho R_2^3}{3\varepsilon_0 r^2}, & r > R_1 \end{cases}$$

To find total energy we will use formula for energy density:

$$u = \frac{\varepsilon_0}{2} E(r)^2$$

Hence, total energy is just integral of u:

$$U = \int u dV = 4\pi \int u r^2 dr = 4\pi \int_0^{R_2} u r^2 dr + 4\pi \int_{R_2}^{R_1} u r^2 dr + 4\pi \int_{R_1}^{\infty} u r^2 dr =$$

$$= const + \frac{\rho^2}{18}(R_1^2/2 - R_2^5/2 - R_2^3 R_1^2 + R_2^5 - \frac{R_2^6}{R_1} + R_2^5) + \frac{\rho^2}{18 R_1}(R_1^3 - R_2^3)^2 =$$

$$= \frac{\rho^2}{18}(R_1^2 - R_2^5/2 - 2R_2^3 R_1^2 + R_2^5 - \frac{2R_2^6}{R_1} + R_2^5) + \frac{\rho^2}{18 R_1}(R_1^3 - R_2^3)^2$$

Here we used the fact, that the const from the first integral is equal to value on the edge, $r = R_1$.

4

Let us first find the potential. We use 1D Poisson equation:

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho}{\varepsilon_0} = -\frac{\rho_0}{\varepsilon_0}\cos\frac{x\pi}{2a}$$

Let us integrate it:

$$\frac{\partial\phi}{\partial x} = -\frac{\rho_0}{\varepsilon_0}\frac{2a}{\pi}\sin\frac{x\pi}{2a} + C$$

$$\phi(x) = \frac{\rho_0}{\varepsilon_0}\left(\frac{2a}{\pi}\right)^2\cos\frac{x\pi}{2a} + Cx + D$$

To find constants C and D we must use that

$$\phi(a) = V_0, \qquad \frac{\partial\phi}{\partial x}(-a) = 0$$

From this one easily finds that

$$C = -\frac{2a\rho_0}{\varepsilon_0\pi}, \qquad D = V_0 + \frac{2a^2\rho_0}{\varepsilon_0\pi}$$

Hence,

$$\phi(x) = \frac{\rho_0}{\varepsilon_0}\left(\frac{2a}{\pi}\right)^2\cos\frac{x\pi}{2a} - \frac{2a\rho_0}{\varepsilon_0\pi}x + \frac{2a^2\rho_0}{\varepsilon_0\pi} + V_0$$

To find field we need to differentiate this:

$$E(x) = \frac{\partial\phi}{\partial x} = -\frac{\rho_0}{\varepsilon_0}\frac{2a}{\pi}\sin\frac{x\pi}{2a} - \frac{2a\rho_0}{\varepsilon_0\pi}$$

5

Potential ϕ is the for all points of the sphere. Hence, we can find it for center (it is easy), and we will know the potential of the sphere. We can write, for O, the center of the sphere

$$\phi = k\frac{q}{l} + \phi'$$

where $k\frac{q}{l}$ is potential created by point charge and ϕ' is potential created by induced charge at the surface of the sphere. However, as the induced charges are equidistant from the center, they potential from them will compensate each other and hence, $\phi' = 0$. Therefore,

$$\phi = k\frac{q}{l} = 9 \cdot 10^9\frac{10^{-9}}{0.1} = 90V$$

Field at the surface at the nearest point will be equal to 0, as the induced charges are distributed in such way, so the external field is screened. Hence, $E = 0$.

6

By definition of electric displacement field and polarization vector we can write

$$\vec{D}(\vec{r}) = \varepsilon\varepsilon_0 E = \varepsilon\varepsilon_0 k\frac{q\vec{r}}{r^3}$$

$$P = D - \varepsilon_0 E = \frac{\varepsilon - 1}{\varepsilon}\frac{q\vec{r}}{r^3}$$

And now we can find the charges:

$$q' = -\int \vec{P}d\vec{S} = \frac{\varepsilon - 1}{\varepsilon}\varepsilon_0 kq\int d\Omega = -\frac{\varepsilon - 1}{\varepsilon}4\pi \cdot \frac{q}{4\pi} = -\frac{\varepsilon - 1}{\varepsilon}q$$

To find volume density you have to divide this by volume of the sphere $4/3\pi r^3$

$$\rho'(r) = \frac{-\frac{\varepsilon-1}{\varepsilon}q}{4/3\pi r^3} = \frac{10^{-9}/2}{4/3\pi r^3}$$

-inside the sphere. Surface density:

$$\sigma = \frac{-\frac{\varepsilon-1}{\varepsilon}q}{4\pi R} = -\frac{10^{-8}/2}{4\pi}$$