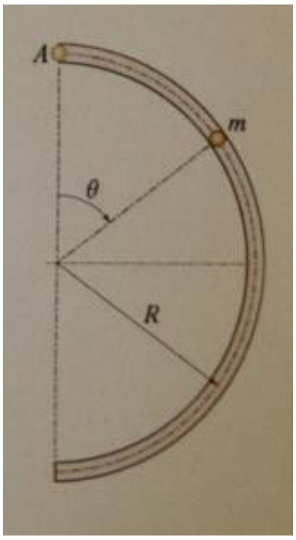
Sample: Mechanics Kinematics Dynamics - Dynamics Assignment

A ball of negligible size and mass m is at rest at A (i.e., $\theta = 0$) in the smooth circular slot that lies in the vertical plane. It is given a small nudge to the right and slides down the slot. Determine the force on the ball due to the slot as a function of the angle θ and evaluate it for $\theta = \pi$.



Solution.

A free body diagram is shown in Figure 1. Introduce a local Cartesian coordinate system (XY) with the center in the center of the ball. The x-axis is directed along the radius of the circle, the y-axis is directed along the tangent to the circle. Write Newton's second law in the projections on the axis:

$$X: \quad ma_n = mg\cos\theta + N;$$
$$Y: \quad ma_\tau = mg\sin\theta,$$

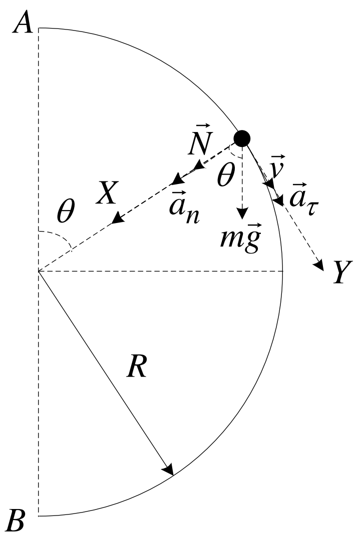


Figure 1.

where N is the force acting on the ball from the slot, $a_n = \frac{v^2}{R}$ is the centripetal acceleration of the ball, $a_\tau = \frac{dv}{dt}$ is the tangential acceleration of the ball, v is the velocity of the ball (the velocity vector $\vec{v}$ tangential to the circumference), g is the acceleration of free fall. Note that from the definition of velocity and the relation between the arc length and the angle it follows:

$$v = \frac{dy}{dt} = \frac{Rd\theta}{dt}.$$

Now we can write the following system of equations:

$$\begin{cases} \frac{mv^2}{R} = mg\cos\theta + N, & (1) \\ \frac{mdv}{dt} = mg\sin\theta, & (2) \\ v = \frac{Rd\theta}{dt} & . \quad (3) \end{cases} \Leftrightarrow \begin{cases} \frac{mv^2}{R} = mg\cos\theta + N, & (1) \\ \frac{dv}{dt} = g\sin\theta, & (2) \\ \frac{d\theta}{dt} = \frac{v}{R}. & (3) \end{cases}$$

To solve this system, divide equation (2) by equation (3):

$$\frac{dv}{dt}\frac{dt}{d\theta} = \frac{gR\sin\theta}{v}; \quad \frac{dv}{d\theta} = \frac{gR\sin\theta}{v}; \quad vdv = gR\sin\theta d\theta.$$

Integrate the last equation:

$$\int vdv = \int gR\sin\theta d\theta; \quad \frac{v^2}{2} = -gR\cos\theta + C,$$

where C is a constant. Find C from the condition that $v = 0$ when $\theta = 0$:

$$C - gR\cos 0 = 0; \quad C - gR = 0; \quad C = gR.$$

So we have:

$$\frac{v^2}{2} = gR - gR\cos\theta; \quad v^2 = 2gR(1-\cos\theta).$$

Substitute this expression into equation (1):

$$\frac{m\cdot 2gR(1-\cos\theta)}{R} = mg\cos\theta + N; \quad 2mg - 2mg\cos\theta = mg\cos\theta + N; \, N(\theta) = mg(2-3\cos\theta).$$

If $N > 0$ then the force N is directed radially towards the center of the circle, if $N < 0$ then the force N is directed radially from the center of the circle. Find N for $\theta = \pi$:

$$N(\pi) = mg(2-3\cos\pi) = 5mg.$$

Check our solution using the law of conservation of energy. Assume that the potential energy of the ball at the point B equals zero. Then the potential energy for the angle θ is $E_p = mgR + mgR\cos\theta = mgR(1+\cos\theta)$.

The kinetic energy is $E_k = \frac{mv^2}{2}$.

According to the energy conservation law:

$$E_k + E_p = const.$$

At the point B

$$E_p = 2mgR,\ E_k = 0.$$

So we have:

$$2mgR = \frac{mv^2}{2} + mgR(1+\cos\theta);\ v^2 = 2gR(1-\cos\theta).$$

Substituting this expression into equation (1), we find $N(\theta) = mg(2-3\cos\theta)$.

Answer: $N(\theta) = mg(2-3\cos\theta);\ N(\pi) = 5mg.$