Sample: Calculus - Derivatives, Integrals, Differential Equations

- *Solve the following differential equations subject to the stated condition on* $y(x)$

  (i) $x^2\frac{dy}{dx} = 1 + y$ with $y(1) = 1$.

  (ii) $x^2\frac{dy}{dx} - 2xy + 3x^2 = 0$ with $y(1) = 2$.

Solution.

(i)

$$x^2\frac{dy}{dx} = 1 + y$$

$$\frac{dy}{y+1} = \frac{dx}{x^2}$$

$$\ln(y+1) = -\frac{1}{x} + C$$

$$y(x) = e^{-\frac{1}{x}+C} - 1 = Ae^{-\frac{1}{x}} - 1$$

$$y(1) = 1$$

$$y(1) = Ae^{-1} - 1 = 1$$

$$A = 2e$$

Finally:

$$y(x) = 2e^{1-\frac{1}{x}} - 1$$

(ii)

$$x^2\frac{dy}{dx} - 2xy + 3x^2 = 0$$

$$y' - \frac{2}{x}y + 3 = 0 \quad (x \neq 0)$$

Now substitute $y(x) = u(x) * v(x)$

$$u'v + (v'u - \frac{2}{x}uv) + 3 = 0$$

$v' - \frac{2}{x}v = 0$ => $\frac{dv}{v} = 2\frac{dx}{x}$ $\ln v = 2\ln x + \ln C$ => $v = Cx^2$

$$u'Cx^2 + 3 = 0$$

$Cdu = -\frac{3dx}{x^2}$ => $u = \frac{3}{Cx} + C^*$

Therefore,

$$y(x) = Cx^2\left(\frac{3}{Cx} + C^*\right) = 3x + C_1x^2$$

$$y(1) = 1$$

$y(1) = 3 + C_1$ => $C_1 = -2$

$$y(x) = 3x - 2x^2$$

- *A city fountain has been polluted with food colouring after a student prank during Prosh Week. The fountain has 500 litres of water within which there is 200 grams of dissolved food colouring. Council workers start to pump water out of the fountain at a rate of 25 litres per minute while at the same time pumping in fresh (food colouring free) water at the same rate. Let $C(t)$ be the amount of food colouring in the fountain t minutes after the workers start to pump water out of the fountain.*
  - *(i) Write down a balance equation for $C(t)$ and the associated initial condition.*
  - *(ii) Derive the solution $C(t) = 200e^{-\frac{t}{20}}$.*
  - *(iii) The fountain is considered "clean" when the food colouring concentration is less than 0.1 grams per litre. How long will it take for the fountain to be clean?*
  - *(iv) Due to a mix up at the depot, the workers discover that the water they are pumping into the fountain has a food colouring concentration of 0.2 grams per litre. Modify (but not solve) the balance equation to account for this.*

Solution.

(i)

$dC = -\frac{25}{500} C\, dt$ and $C(0) = 200$

(ii)

$$\frac{dC}{C} = -\frac{dt}{20}$$

$$\ln C = -\frac{t}{20} + A$$

$$C = e^{\frac{t}{20}+A} = Be^{-\frac{t}{20}}$$

$C(0) = 200$ => $C(0) = B$ => $B = 200$

Finally:

$$C(t) = 200e^{-\frac{t}{20}}$$

(iii)

Concentration equals:

$$c = \frac{C}{V} = \frac{200e^{-\frac{t}{20}}}{500\ l} = 0.4e^{-\frac{t}{20}} = 0.1$$

$$e^{\frac{t}{20}} = 4$$

$$t = 20 \ln 4 = 40 \ln 2 = 27.7\ m$$

(iv)

$$dC = \left(-\frac{25}{500}C + 0.2 * 25\right) dt$$

- *Find the derivative for each of the following functions.*
  - *(i)* $f(x) = 2\ln(\sin 3x)$
  - *(ii)* $f(x) = e^{4\cos x}$
  - *(iii)* $f(x) = \arcsin\sqrt{x+1}$
  - *(iv)* $f(x) = \ln(\arctan x)$

Solution.

(i) $f'(x) = 2\frac{1}{\sin 3x}3\cos 3x \; = 6ctg(3x)$

(ii) $f'(x) = e^{4\cos x}4(-\sin x) = -4e^{4\cos x}\sin x$

(iii) $f'(x) = \frac{1}{\sqrt{1-(\sqrt{x+1})^2}}\frac{1}{2\sqrt{x+1}} = \frac{1}{\sqrt{1-|x+1|}}\frac{1}{2\sqrt{x+1}} = \frac{1}{2\sqrt{-x(x+1)}}$

(iv) $f'(x) = \frac{1}{\arctan x}\frac{1}{(1+x^2)}$

- *Find the following integrals.*
  - *(i)* $\int_2^4 3x\cos x^2\, dx$
  - *(ii)* $\int \frac{\cos x}{1+\sin^2 x} dx$

Solution.

(i)

$$\int_2^4 3x\cos x^2\, dx = \int_2^4 \frac{3}{2}\cos x^2\, d(x^2) = \frac{3}{2}\sin x^2 \,|_2^4 = \frac{3}{2}(\sin 16 - \sin 4)$$

(ii)

$$\int \frac{\cos x}{1+\sin^2 x} dx = \int \frac{d(\sin x)}{1+\sin^2 x} = \arctan(\sin x) \; + C$$