Sample: Graph Theory - Cycles

2. (a) Let N be actual number of edges used to bound regions in graph G and r be actual number of regions in G. Then we have at least 4 edge cycles, each region would be bounded by at least 4 edge.

$$4r \leq N,$$

each edge is shared by two regions

$$N \leq 2e,$$

$$4r \leq N \leq 2e$$

Using Euler's formula: $n - e + f \ = 2$ (where n is number of vertices, e is a number of edges, and f is a number of faces) we obtain

$$4(e - n + 2) \leq 2e,$$

$$4e - 4n + 8 - 2e \leq 0,$$

$$2e \leq 4n - 8,$$

$$e \leq 2n - 4$$

QED.

(b) Previous statement holds for any simple, connected planar graph. Verify that this condition holds for $K_{3,3}$. This graph is simple, connected and has no 3 edges cycles. It has 6 vertices and 9 edges. If it is planar then should hold the following

$$9 \leq 2 * 6 - 4 = 12 - 4 = 8$$

But it isn't true. We get the contradiction. Therefore, $K_{3,3}$ is not planar.

3. (a) Let us draw the graph in which each vertex is a point on the plane, two vertices is connected if distance between them 1cm. If we consider the graphs with maximum such pairs than at most degree of each vertex is 3 (draw an unit circle with center at one of the point and try to find another points on this circle that satisfy all conditions), and this graph is planar. The number of edges on this graph is the number of pairs that are in distance 1cm.

Let us prove one important result: If G is planar graph then it satisfies the condition $e \leq 3n - 6$.

Let N be actual number of edges used to bound regions in graph G and r be actual number of regions in G, then we have at least 3 edge cycles, each region would be bounded by at least 3 edge.

$$3r \leq N$$

each edge is shared by two regions

$$N <= 2e,$$

$$3r <= N <= 2e$$

using $r = e - v + 2$ we get

$$3(e - v + 2) <= 2e$$

$$e <= 3v - 6.$$

According to this result we obtain the statement of our task.

(b) The equality in (a) is reach only when n=3. Because if n=4 we obtain maximum 5 edges (3*4-6=6¿5). For other cases it was a subgraphs (like a previous cases $n \leq 4$) which is connected with one edge or disconnected. Look at the figure.

4. We will use the mathematical induction on the number of vertices.

Let m is a number os leafs

Base: for n=2 our tree is a path and $\delta(T) = 1$ and number of leafs m is 2. The statement is hold.

Step: Suppose that for n=k we have inequality $m \geq \delta(T)$.

Let us add one vertex. This vertex is a leaf because we can connect it only with one other vertex, and we get the graph T'. If we connect it to a vertex that has maximum degree then $\delta(T') = \delta(T) + 1$ and $m' = m + 1$. We obtain

$$\delta(T') = \delta(T) + 1 \leq m + 1 = m'$$

or

$$\delta(T') \leq= m'$$

The statement holds.

If we connect it to any other vertex then $\delta(T') = \delta(T)$ and $m' = m + 1$ and

$$\delta(T') = \delta(T) < m + 1 = m'$$

QED.

5. If the graph is 2-connected than any vertex and any adge belong to the simple cycle. From each vertex we can see at least two neighboring vertices, so if we place camera at the first vertex than we shouldn't place the camera at the neighboring. Therefore we can place the cameras through one vertex. If the cycle is odd then the last two vertices will be without cameras. Penultimate vertex we can see from previous camera and the last one we can see from the first camera. Thus we get at most $[\frac{n}{2}]$ cameras.

The museum with one triangle room has 3 vertex and need exactly $[\frac{3}{2}] = 1$ camera.

6. Lets the disk is a vertex and if two vertices is connected then corresponding disks are tangent to each other. consider two vertices that connected. Both of them can be connected only with two another vertices (only two circles can be tangent to considering circles which are tangent). This four vertices we color in four different colors. If we consider any another pair of vertices we get the same situation. As we know four circle can not touches to each other in pairs.

7. Let $G = (V, E)$ be a graph satisfying the properties in the problem statement. If G has no odd cycles, it is bipartite graph and $\chi(G) \leq 2 < 5$. So

let C be a shortest odd cycle in G. This is actually an induced subgraph of G, because if it had a chord, the chord together with one of the two halfs of the cycle would form a smallest odd cycle.

The subgraph $G' = G(V - V(C))$ does not contain any odd cycles, as such a cycle would not intersect C in G. So G' is bipartite and can be colored with 2 colors. The cycle C can be colored with 3 colors. These colorings can be combined to a valid 5-coloring of G.

The chromatic number of K_5 is 5.

8. Consider arbitrary tree with n vertices. Try to draw it on our set P with straight line edges which are not crossing. In our mind counting the points of P in order from biggest y-coordinate to smaller and denote them as $v_1, v_2, ..., v_n$. Let v_1 will be a root of our tree. And the root of tree is connected with n_1 other vertices. Connect our root with points $v_2, ..., v_{n_1+1}$. From this set $(v_2, ..., v_{n_1+1})$ we take the vertex with minimum x-coordinate and going from the top through the left side and draw the longest branch in the same manner. Obviously that we can do it without crossing any edge. All vertices which leaves take place on the right side from this branch. Similarly we take the next branch from the given tree and repeated with it the same steps. Therefore, we obtain a graph which is similar to the given, has straight line edges that are not crossing.

QED.

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