Sample: Real Analysis - Continuity and Differentiability

1)

Since $\lim_{x\to\infty} f(x)$ exists, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^x}{e^x} = (using\ l'Hospital\ rule) = \lim_{x\to\infty} \frac{(f(x)e^x)'}{(e^x)'}$$
$$= \lim_{x\to\infty} f(x) + f'(x)$$

So we get: in case $\lim_{x\to\infty} f'(x)$ exists, it equals to 0.

Inductively continuing this statement, we get that the same holds for $f'', f''', \ldots$

Since $\lim_{x\to\infty} f^{(k)}(x)$ exists, we get that

$$\lim_{x\to\infty} f^{(k)}(x) = 0$$

Inductively getting back to lower derivatives we get:

$$\lim_{x\to\infty} f^{(i)}(x) = 0$$

for $i = 1,2, \ldots, k$

So the statement is proved.

2)

Consider a function

$$D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$D(x)$ is a linear combination of functions $f(x), g(x), h(x)$.

Since f, g, h are continuous on $[a, b]$, $D(x)$ is continuous on $[a, b]$.

Since f, g, h are differentiable on (a, b), $D(x)$ is differentiable on (a, b).

$$D(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

$$D(b) = \begin{vmatrix} f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Last 2 equalities hold because matrices the determinant is taken has 2 identical rows.

By Rolle's theorem,

$$\exists c \in (a,b): D'(c) = 0$$

$$'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

The statement is proved.

3)

Consider a function

$$F(x) = \frac{1}{\frac{b-a}{2}}\left(f\left(x + \frac{b-a}{2}\right) - f(x)\right)$$

It is defined at $\left[a, \frac{a+b}{2}\right]$, continuous at $\left[a, \frac{a+b}{2}\right]$ (because f is continuous) and twice-differentiable on $\left(a, \frac{a+b}{2}\right)$. By mean value theorem,

$$\exists d \in \left(a, \frac{a+b}{2}\right): F'(d) = \frac{F\left(\frac{a+b}{2}\right) - F(a)}{\frac{b-a}{2}}$$

$$\frac{F\left(\frac{a+b}{2}\right) - F(a)}{\frac{b-a}{2}} = \frac{\frac{1}{\frac{b-a}{2}}\left(f(b) - f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) + f(a)\right)}{\frac{b-a}{2}}$$

$$= \frac{f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)}{\frac{(b-a)^2}{4}}$$

$$F'(d) = \frac{1}{\frac{b-a}{2}}\left(f'\left(d + \frac{b-a}{2}\right) - f'(d)\right)$$

Consider a function f' on $\left[d, d + \frac{b-a}{2}\right]$. It is continuous on this interval and differentiable on $\left(d, d + \frac{b-a}{2}\right)$ (because f is twice differentiable). Thus, by mean value theorem,

$$\exists c \in \left(d, d + \frac{b-a}{2}\right): f''(c) = \frac{f'\left(d + \frac{b-a}{2}\right) - f'(d)}{\frac{b-a}{2}}$$

Substituting this into expression we got earlier we get:

$$f''(c) = \frac{f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)}{\frac{(b-a)^2}{4}}$$

$$f''(c) \cdot \frac{(b-a)^2}{4} = f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)$$

Since $c \in \left(d, d + \frac{b-a}{2}\right) \subset (a, b)$ we proved the needed statement.

1)

f is continuous at $[a, b]$.

$$\int_a^b f^2(x)dx = 0$$

Suppose

$$f(c) \neq 0$$

for some $c \in [a, b]$.

If we take $\varepsilon = \frac{|f(c)|}{2}$ in the definition of continuity of f at point c we get:

$$\exists \delta: x \in (c - \delta, c + \delta) \cap (a, b): |f(x) - f(c)| < \frac{|f(c)|}{2}$$

$$-\frac{|f(c)|}{2} < f(x) - f(c) < \frac{|f(c)|}{2}$$

So

$$|f(x)| > \frac{|f(c)|}{2}$$

at in the interval $(c - \delta, c + \delta) \cap (a, b)$.

Then

$$\int_a^b {}^2(x)dx \geq \int_{(c-\delta,c+\delta)\cap(a,b)} {}^2(x)dx \geq \int_{(c-\delta,c+\delta)\cap(a,b)} \frac{|f(c)|}{2}dx$$
$$= \frac{|f(c)|}{2} \cdot (\min(c+\delta,b) - \max(c-\delta,a)) > 0$$

So we get contradiction with

$$\int_a^b {}^2(x)dx = 0$$

So $f(x) = 0$ at $[a,b]$.