Sample: Topology - Cantor Set

B2

$$f(a_1, a_2, \dots) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

Note, that map f represents ternary representation of real numbers from the interval $[0,1]$. Really, in case ternary representation of a number x from $[0,1]$ is $0.a_1a_2 \dots$ then

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

In case the sequence $\{a_i\}$ have all twos starting from some index j, then

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{j} \frac{a_n}{3^n} + \sum_{n=j+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{j} \frac{a_n}{3^n} + \frac{1}{3^j}$$

So the sequence $\{a_1, a_2, \dots, a_j, 2,2, \dots\}$ is mapped to the value with finite ternary representation. The same holds for the sequences ending by zeroes.

On the other hand, in case number x has finite ternary representation, it can be represented using sequences $\{a_i\}$ only in two ways:with ending 0-s or ending 2- s.

So we need to prove that endpoints of intervals from C_n have finite ternary representation.

Let's prove this by induction.

In case $n = 0$ we have 2 end points: 0 and 1.

$$0 = 0.00 \dots$$

$$1 = 0.22 \dots$$

and the statement holds.

In case $n = k + 1$ we form new intervals in such a way.

Suppose we have interval (a, b). Then we change it into 2: $\left(a, \frac{2}{3}a + \frac{1}{3}b\right)$ and $\left(\frac{1}{3}a + \frac{2}{3}b, b\right)$.By induction assumption, points a and b have finite ternary representation of length at most N. Then linear combinations $\frac{2}{3}a + \frac{1}{3}b$ and $\frac{1}{3}a + \frac{2}{3}b$ have ternary representation of length at most $N + 1$.

So the statement is proved.

B3

Consider any number x from the Cantor set. Then

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

and $a_i \in \{0,2\}$.

Consider a sequence $\{b_n\}$ with elements

$$b_n = \begin{cases} 0, a_n = 0 \\ 1, a_n = 2 \end{cases}$$

So we built a map from Cantor set to set of sequences that consist of zeros and ones. This map is bijective, because it has inverse:

$$a_n = \begin{cases} 0, b_n = 0 \\ 2, b_n = 1 \end{cases}$$

Since set of sequences consisting from 0s and 1s can be represented as

$$\prod_{n \in \mathbb{N}} D$$

, we have just proved that Cantor set and $\prod_{n \in \mathbb{N}} D$ are homeomorphic.