Sample: Topology - Applied Topology

Task 1 .

Give an explicit formula for a homeomorphism from a square to a circle in the Euclidean plane.

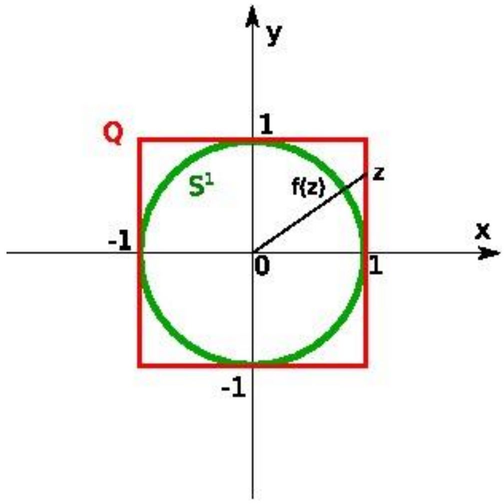
Proof. Let

$$S^1 = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$$

be the unit circle in $\mathbf{R}^2$, and

$$Q = \partial([0,1] \times [0,1]) = \{(x, y) \in \mathbf{R}^2 | \max\{|x|, |y|\} = 1.$$

be the square with side 2 in the Euclidean plane centered at the origin, see Figure:



Define the map

$$f: Q \to S^1$$

by the formula:

$$f(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

Since

$$\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = \frac{x^2+y^2}{x^2+y^2} = 1,$$

we see that $f(Q) \subset S^1$.

We claim that f is homeomorphism.

Geometrically, f acts as follows. Let $z \in Q$. Consider the straight segment $[O, z]$ connecting the origin $O = (0,0)$ with the point z. This segment intersect S^1 at a unique point, and this point is denoted by $f(z)$, see Figure above.

To show that f is a homeomorphism it suffices to show that f is a continuous bijection whose inverse is also continuous.

1) $\boldsymbol{f}$ **is continuous.** Notice that the same formula for f defines the map

$$F: \mathbf{R}^2 \backslash 0 \to S^1$$

by the formula:

$$F(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

This map is continuous, since $\sqrt{x^2+y^2} \neq 0$ for all $(x,y) \neq (0,0)$.
But f is the restriction of F onto Q, and so f is continuous.

2) $\boldsymbol{f}$ **is a bijection.** For $z=(x,y) \neq 0$ let
$$R_z = \{tz = (tx,ty) \in \mathbf{R}^2 | t>0\}$$
be the open ray starting at the origin and passing through z.

Lemma. *For any $z \neq 0$, each of the following intersections*
$$R_z \cap Q, R_z \cap S^1$$
consists of a unique point.

This lemma implies that f is a bijection.
Indeed, first notice that if $z \in Q$, then
$$z = R_z \cap Q,$$
and by geometrical description of f
$$f(z) = R_z \cap S^1.$$
Now if $z_1 \neq z_2 \in Q$, then the rays $R_{z_1} \cap R_{z_2} = \emptyset$, and so
$$f(z_1) = R_{z_1} \cap S^1 \neq R_{z_2} \cap S^1 \neq f(z_2).$$
Thus $\boldsymbol{f}$ **is injective**.
Moreover, let $p \in S^1$ and
$$z = R_p \cap Q,$$
then $p = f(z)$, and so $\boldsymbol{f}$ **is surjective.**

Proof of Lemma. Suppose $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2) \in R_z \cap S^1$. Then
$$x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1,$$
and
$$z_1 = (x_1,y_1) = t_1 z_1 = (t_1 x, t_1 y), z_2 = (x_2,y_2) = t_2 z_2 = (t_2 x, t_2 y),$$
that is
$$x_1 = t_1 x, y_1 = t_1 y, x_2 = t_2 x, y_2 = t_2 y.$$
Then for $i = 1,2$
$$x_i^2 + y_i^2 = t_i^2 x^2 + t_i^2 y^2 = t_i^2(x^2+y^2) = 1,$$
so
$$t_1^2(x^2+y^2) = t_2^2(x^2+y^2) = 1.$$
Since $z \neq 0$, we have that $x^2+y^2 \neq 0$, and so
$$t_1^2 = t_2^2.$$
But $t_1, t_2 > 0$, whence
$$t_1 = t_2,$$
and thus $z_1 = z_2$.
Similarly, suppose $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2) \in R_z \cap Q$. Thus
$$\max\{|x_1|,|y_1|\} = \max\{|x_2|,|y_2|\} = 1,$$
and
$$z_1 = (x_1,y_1) = t_1 z_1 = (t_1 x, t_1 y), z_2 = (x_2,y_2) = t_2 z_2 = (t_2 x, t_2 y),$$
that is
$$x_1 = t_1 x, y_1 = t_1 y, x_2 = t_2 x, y_2 = t_2 y.$$

Then for $i = 1,2$

$$\max\{|x_i|, |y_i|\} = \max\{|t_i x|, |t_i, y|\} = t_i \max\{|x|, |y|\} = 1,$$

since $t_i > 0$. Thus

$$t_1 \max\{|x|, |y|\} = t_2 \max\{|x|, |y|\} = 1.$$

As $z \neq 0$, we have that $\max\{|x|, |y|\} > 0$, whence

$$t_1 = t_2,$$

and thus $z_1 = z_2$. Lemma is proved.

3) $\boldsymbol{f^{-1}}$ **is continuous.** By 1) and 2) f is a continuous bijection. Since Q is compact, and S^1 is Hausdorff, it follows that the inverse of f is also continuous, and so f is a homeomorphism.

Task 2 .

Let $D^\circ = \{(x,y) \in \mathbf{R}^2 : x^2+y^2<1\}$ be the open disk.

(a) Let $f: D^\circ \to \mathbf{R}^3$ be defined by $f((r,\theta)) = (r,\theta,r)$ where (r,θ) and (r,θ,r) are in polar and cylindrical coordinates, respectively. Show that f is an embedding of D° and describe the embedded copy $f(D^\circ)$ in $\mathbf{R}^3$.

(b) Describe the stereographic projection in $\mathbf{R}^3$ of $f(D^\circ)$ onto the plane $z=0$ from a light source located at the point $(0,0,1)$. Is this a homeomorphism? Find an explicit formula for this stereographic projection function p.

(c) Describe all planes in $\mathbf{R}^3$ for which stereographic projection of $f(D^\circ)$ from a light source at $(0,0,1)$ is a homeomorphism.

Solution.

(a) Notice that in Cartesian coordinates (x,y) we have that $r=\sqrt{x^2+y^2}$. Then the map $f: D^\circ \to \mathbf{R}^3$

$$f((r,\theta)) = (r,\theta,r)$$

in coordinates (x,y) is given by the formula:

$$f(x,y) = (x,y,\sqrt{x^2+y^2}).$$

This formula shows that f $\boldsymbol{f}$ **is continuous**.

Moreover $\boldsymbol{f}$ **is injective**, since for $(r,\theta) \neq (r',\theta') \in D^\circ$, we have that

$$f((r,\theta)) = (r,\theta,r) \neq (r',\theta',r') = f((r,\theta)).$$

Also consider the map $p: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $p(x,y,z) = (x,y)$. Then

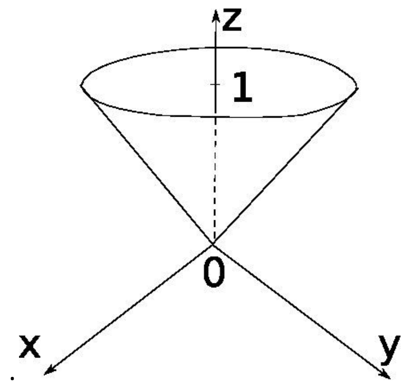
$$p \circ f(x,y) = p(x,y,\sqrt{x^2+y^2}) = (x,y)$$

and so the restriction of p onto $f(D^\circ)$ is the inverse to f. This proves that f is an embedding, i.e. a homeomorphism onto its image.

Notice $f(D^\circ)$ is the graph of the function $g(x,y) = \sqrt{x^2+y^2}$, and so it is the open cone over the circle

$$\{x^2+y^2=1, z=1\}$$

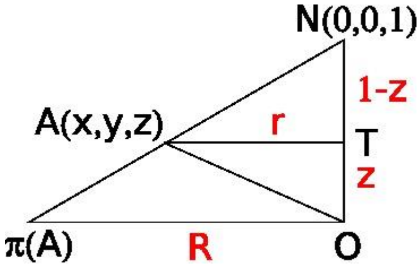
and vertex at the origin



(b) Let us find formula for stereographic projection π from the point $N=(0,0,1)$. This maps associates to each point $A(x,y,z) \in \mathbf{R}^3$ the intersection point of the ray NA with the plane $\{z=0\}$. In particular, this map is defined only for points A with $z<1$, i.e on the open half-space

$$X = \{(x,y,z) | z<1\}.$$

Notice that this projection preserves θ, that is $\theta(A) = \theta(\pi(A))$.
Consider the figure:



It follows from similarity of triangles ΔATN and $\pi(A)ON$ that

$$\frac{R}{r} = \frac{1}{1-z},$$

whence π is given by the formula

$$\pi(r, \theta, z) = \left(\frac{r}{1-z}, \theta\right).$$

Then the composition $\pi \circ f: D^{\circ} \to \mathbf{R}^2$ is given by the formula

$$\pi \circ f(r, \theta) = \pi(r, \theta, r)\left(\frac{r}{1-r}, \theta\right).$$

Notice that the function

$$y(r) = \frac{r}{1-r} = \frac{r-1+1}{1-r} = \frac{1}{1-r} - 1$$

is strictly monotone and maps the interval $[0,1)$ onto $[0, +\infty)$. Its inverse is given by

$$y^{-1}(\rho) = 1 - \frac{1}{1+\rho} = \frac{1+\rho-1}{1+\rho} = \frac{\rho}{1+\rho} =$$

This implies that $\pi \circ f$ is a homeomorphism and its inverse is given by

$$(\pi \circ f)^{-1}(r, \theta) = \left(\frac{r}{1+r}, \theta\right).$$

(c) Let P be a plane in $\mathbf{R}^3$ such that π is defined on P. As noted in 2) the stereographic projection $\pi: X \to \mathbf{R}^2$ is defined only for points (x, y, z) with $z < 1$, whence $P \subset X$. But this is possible only if P is given by the equation $z = c$, where $c < 1$.
We claim that for every such plane the restriction of π to P is a homeomorphism $P \to \mathbf{R}^2$. Let $(r, \phi, c) \in P$. Then

$$\pi(r, \theta, c) = \left(\frac{r}{1-c}, \theta\right).$$

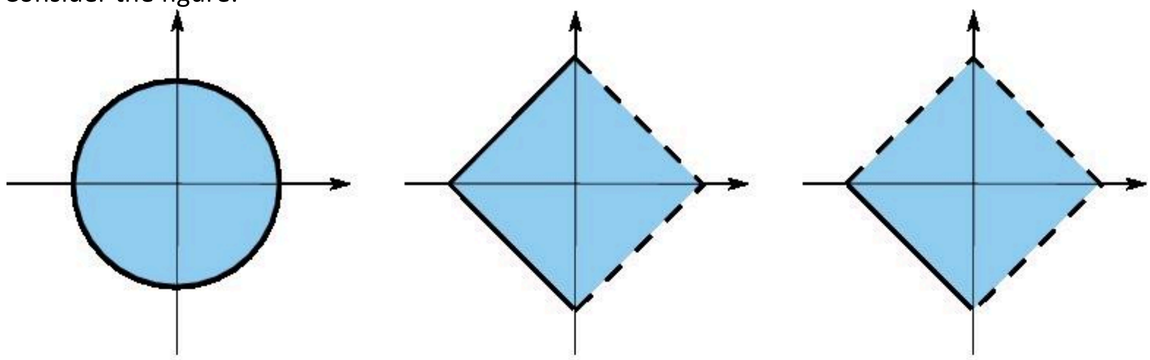
Therefore in Cartesian coordinates we have that

$$\pi(x, y, c) = \left(\frac{x}{1-c}, \frac{y}{1-c}\right),$$

that is the restriction $\pi|_P$ of π to P is a homothety with coefficient $\frac{1}{1-c}$. Hence its inverse is a homothety with coefficient $1 - c$, and so $\pi|_P$ is a homeomorphism.

Task 3 .

Consider the figure:



(a) (b) (c)

Are these sets open, closed, both, or neither in $\mathbf{R}\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}_l$, where $\mathbf{R}_l$ is the plane with lower limit topology?

Solution.

Recall that lower limit topology on $\mathbf{R}$ is generated by half-open intervals

$$\{[a,b)\subset\mathbf{R}|a<b\}.$$

The set $\mathbf{R}$ with this topology is denoted by $\mathbf{R}_l$.

Notice that every interval of the form (a,b) is open in $\mathbf{R}_l$, since

$$(a,b)=\cup_{i\geq 1}\ [a-1/i,b)$$

Therefore every closed (resp. open) subset of $\mathbf{R}$ is also closed (resp. open) in $\mathbf{R}_l$, and so topology on $\mathbf{R}_l$ is stronger than the usual topology on $\mathbf{R}$.

This implies that the topology on $\mathbf{R}_l\times\mathbf{R}_l$ is stronger than the topology of $\mathbf{R}_l\times\mathbf{R}$, which in turn is stronger than topology on $\mathbf{R}^2=\mathbf{R}\times\mathbf{R}$.

Also notice that topology on $\mathbf{R}\times\mathbf{R}$ is generated by sets of the form

$$\{(a,b)\times(c,d)|a<b,c<d\},$$

topology on $\mathbf{R}_l\times\mathbf{R}$ is generated by sets of the form

$$\{[a,b)\times(c,d)|a<b,c<d\},$$

and topology on $\mathbf{R}_l\times\mathbf{R}_l$ is generated by sets of the form

$$\{[a,b)\times[c,d)|a<b,c<d\},$$

see figure:

(a) (b) (c)

(a) The set in (a) is a closed 2-disk and can be defined by the formula:

$$A=\{(x,y)\in\mathbf{R}^2|x^2+y^2\leq 1.$$

Let us show that A is **closed**. Let $\{(x_i,y_i)\}_{i\geq 1}\subset A$ be a sequence of points converging to some $(x,y)\in\mathbf{R}^2$. Then

$$x^2+y^2=\lim_{i\to\infty}x_i^2+y_i^2\leq 1,$$

whence $(x, y) \in A$, and so A is closed.
Therefore A is also closed in $\mathbf{R}_l \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}_l$.
On the other hand, **A is not open in neither of $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}_l$.**
Since $\mathbf{R}_l \times \mathbf{R}_l$ has the strongest topology, it suffices to prove that A is not open in $\mathbf{R}_l \times \mathbf{R}_l$.
Consider the point $z = (1,0) \in A$ and let $U = [a, b) \times [c, d)$ be any neighborhood of z in $\mathbf{R}_l \times \mathbf{R}_l$, so $a \leq 1 < b$ and $c \leq 0 < d$. Take any $q \in (1, b)$. Then the point $(q, 0) \in U$, but $q \notin A$. Thus $U \not\subset A$, and so A is not open in $\mathbf{R}_l \times \mathbf{R}_l$. Therefore it is not open in $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$.

(b) The set in (b) can be given by the formula:

$$B = \{(x, y) \in \mathbf{R}^2 | |x| + |y| \leq 1\} \backslash \{(x, y) | |x| + |y| = 1 \; and \; y > 0\}.$$

This set is neither open nor closed in any of topologies of $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}_l$.
Again it suffices to show this for $\mathbf{R}_l \times \mathbf{R}_l$.
Let us show that B is not open in $\mathbf{R}_l \times \mathbf{R}_l$. Consider the point $(-1,0) \in B$ and let $U = [a, b) \times [c, d)$ be any neighborhood of z in $\mathbf{R}_l \times \mathbf{R}_l$, so $a \leq -1 < b$ and $c \leq 0 < d$. Take any $q \in (0, d)$. Then the point $(-1, q) \in U$, but $q \notin B$, since

$$|-1| + |q| > 1.$$

Thus $U \not\subset B$, and so B is not open in $\mathbf{R}_l \times \mathbf{R}_l$. Therefore it is not open in $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$.
Let us show that B is not closed in $\mathbf{R}_l \times \mathbf{R}_l$. For this it suffices to show that $\mathbf{R}_l \times \mathbf{R}_l \backslash B$ is not open. Again consider the point $(0.5, -0.5) \in B$ and let $U = [a, b) \times [c, d)$ be any neighborhood of z in $\mathbf{R}_l \times \mathbf{R}_l$, so $a \leq 0.5 < b$ and $c \leq -0.5 < d$. Take any $q \in (0.5, b)$. Then the point $(q, -0.5) \in U$, but $q \notin B$, since

$$|q| + |-0.5| > |0.5| + |-0.5| = 1.$$

Thus $U \not\subset \mathbf{R}_l \times \mathbf{R}_l \backslash B$, and so $\mathbf{R}_l \times \mathbf{R}_l \backslash B$ is not open in $\mathbf{R}_l \times \mathbf{R}_l$, whence B is not closed in $\mathbf{R}_l \times \mathbf{R}_l$ Therefore it is not open in $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$ as well.

(c) The set in (c) can be given by the formula:

$$C = \{(x, y) \in \mathbf{R}^2 | |x| + |y| < 1\} \cup \{(x, y) | |x| + |y| = 1 \; and \; x + y = -1\}.$$

We claim that this set is also neither open nor closed in any of topologies of $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}_l$.
Again it suffices to show this for $\mathbf{R}_l \times \mathbf{R}_l$.
Let us show that C is not open in $\mathbf{R}_l \times \mathbf{R}_l$. Consider the point $(-1,0) \in C$ and let $U = [a, b) \times [c, d)$ be any neighborhood of z in $\mathbf{R}_l \times \mathbf{R}_l$, so $a \leq -1 < b$ and $c \leq 0 < d$. Take any $q \in (0, d)$. Then the point $(-1, q) \in U$, but $q \notin C$, since

$$|-1| + |q| > 1.$$

Thus $U \not\subset C$, and so C is not open in $\mathbf{R}_l \times \mathbf{R}_l$. Therefore it is not open in $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$.
Let us show that C is not closed in $\mathbf{R}_l \times \mathbf{R}_l$. For this it suffices to show that $\mathbf{R}_l \times \mathbf{R}_l \backslash C$ is not open. Again consider the point $(0.5, -0.5) \in C$ and let $U = [a, b) \times [c, d)$ be any neighborhood of z in $\mathbf{R}_l \times \mathbf{R}_l$, so $a \leq 0.5 < b$ and $c \leq -0.5 < d$. Take any $q \in (0.5, b)$. Then the point $(q, -0.5) \in U$, but $q \notin C$, since

$$|q| + |-0.5| > |0.5| + |-0.5| = 1.$$

Thus $U \not\subset \mathbf{R}_l \times \mathbf{R}_l \backslash C$, and so $\mathbf{R}_l \times \mathbf{R}_l \backslash C$ is not open in $\mathbf{R}_l \times \mathbf{R}_l$, whence C is not closed in $\mathbf{R}_l \times \mathbf{R}_l$ Therefore it is not open in $\mathbf{R} \times \mathbf{R}$, $\mathbf{R}_l \times \mathbf{R}$ as well.

Answer. In the case (a) the set A is closed in each spaces $\mathbf{R}\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}_l$ but not open in neither of the spaces.
In the cases (b) and (c) the corresponding sets are neither open nor closed in any of the spaces $\mathbf{R}\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}$, $\mathbf{R}_l\times\mathbf{R}_l$.