Sample: Complex Analysis - Singularities

1. Identify singularities of the function $f(z) = \frac{z}{\sin(z^2)}$ and classify them. Find the residues of the function at those points.

Solution:

Singularities of the function $f(z) = \frac{z}{\sin(z^2)}$ are solutions to the equation $\sin(z^2) = 0$ and ∞.

$z^2 = \pi n, n \in \mathbb{Z}$

$z_1 = \pm\sqrt{\pi n}, n \in \mathbb{N}; z_2 = \pm i\sqrt{\pi(-n)}, n \in \mathbb{N}_-; z_3 = 0, n = 0.$

As we see, ∞ is not isolated singularity, because at any neighborhood there are infinite set of singular points such as z_1 and z_2.

To find residues at poles z_1 and z_2, use the theorem: if $f(z) = \frac{\phi(z)}{\psi(z)}$ is analytical at ring with center at point a and radii r1=0 and r2= ε, and $\phi(z), \psi(z)$ are analytical at circle with center at point a and radius r= ε, $\phi(a) \neq 0, \psi(a) = 0, \psi'(z) \neq 0$, then a is a pole of order 1 and

$$Res(f(z), a) = Res\left(\frac{\phi(z)}{\psi(z)}, a\right) = \frac{\phi(a)}{\psi'(a)}$$

$\phi(z) = z, \psi(z) = \sin(z^2), \psi'(z) = 2z\cos(z^2)$. As we see $\phi(z_{1,2}) \neq 0$, $\psi'(z_{1,2}) = 2z_{1,2}\cos\pi n = 2z_{1,2}(-1)^n \neq 0$ and we can use the theorem:

$$Res(f(z), z_{1,2}) = \frac{z_{1,2}}{2z_{1,2}(-1)^n} = \frac{1}{2}(-1)^n$$

Maclaurin series of sin(x) is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

That's why f(z) at z=0 behaves as 1/z because

$$f(z) = \frac{z}{\sin z^2} = \frac{z}{z^2 - \frac{z^{2\cdot 3}}{3!} + \frac{z^{2\cdot 5}}{5!} - \cdots} = \frac{z}{z^2(1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots)} = \frac{1}{z} + o(1)$$

It means that in Laurent series $a_{-1} = 1$ and 0 is a pole of order 1, thus $Res(f(z), 0) = a_{-1} = 1$.

Answer:

∞ is not isolated singularity;

$\pm\sqrt{\pi n}, n \in \mathbb{Z}\backslash\{0\}$ are poles of order 1, $Res(f(z), \sqrt{\pi n}) = \frac{1}{2}(-1)^n$

$\pm i\sqrt{\pi n}, n \in \mathbb{Z}\backslash\{0\}$ are poles of order 1, $Res(f(z), \sqrt{\pi n}) = \frac{1}{2}(-1)^n$

0 is a pole of order 1, $Res(f(z), 0) = 1$

2. Evaluate the following integrals and justify your calculation.

a) $\int_{-2\pi}^{0} \frac{dx}{4\sin x + 5}$

Solution:

We know a theorem, if R(x,y) is a rational function, then

$$\int_0^{2\pi} R(\cos\phi, \sin\phi) d\phi = 2\pi \sum_{|z_k|<1} Res[\frac{1}{z} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right), z_k]$$

Transform the integral using substitution x=-y, dx=-dy

$$\int_{-2\pi}^{0}\frac{dx}{4\sin x+5}=\int_{2\pi}^{0}\frac{-dy}{-4\sin y+5}=\int_{0}^{2\pi}\frac{dx}{5-4\sin x}$$

$$=2\pi\sum_{|z_k|<1}Res[\frac{1}{z}\cdot\frac{1}{5-4\frac{1}{2i}\left(z-\frac{1}{z}\right)},z_k]$$

$$=2\pi\sum_{|z_k|<1}Res[\frac{1}{5z+2iz^2-2i},z_k]$$

Do polynomial factorization of a denominator $2iz^2+5z-2i$. Solve a quadratic equation

$2iz^2+5z-2i=0$

$D=25-4\cdot(2i)(-2i)=25-16=9$

$z_{1,2}=\frac{-5\pm\sqrt{9}}{2\cdot 2i}=\frac{-5\pm 3}{4i},z_1=-\frac{8}{4i}=2i,z_2=-\frac{2}{4i}=\frac{1}{2}i$

$2iz^2+5z-2i=2i(z-2i)(z-\frac{1}{2}i)$

Only $|z_2|$=0.5<1, so $2\pi\sum_{|z_k|<1}Res[\frac{1}{5z+2iz^2-2i},z_k]=2\pi Res[\frac{1}{2i(z-2i)\left(z-\frac{1}{2}i\right)},\frac{1}{2}i]$

At a simple pole c, the residue of f is given by:

$Res(f,c)=\lim_{z\to c}(z-c)f(z)$

z_2 is a simple pole, thus

$$Res\left[\frac{1}{2i(z-2i)\left(z-\frac{1}{2}i\right)},\frac{1}{2}i\right]=\lim_{z\to\frac{1}{2}i}\left(z-\frac{1}{2}i\right)\frac{1}{2i(z-2i)\left(z-\frac{1}{2}i\right)}=\lim_{z\to\frac{1}{2}i}\frac{1}{2i(z-2i)}$$

$$=\frac{1}{2i\left(\frac{1}{2}i-2i\right)}=\frac{1}{-2i\cdot\frac{3}{2}i}=\frac{1}{3}$$

At last, $2\pi Res\left[\frac{1}{2i(z-2i)\left(z-\frac{1}{2}i\right)},\frac{1}{2}i\right]=\frac{2\pi}{3}$

Answer: $\frac{2\pi}{3}$.

b) $\int_{-\infty}^{\infty}\frac{z^2}{(z^2-6z+10)^2}dx$

Solution:

Do polynomial factorization of a denominator $z^2-6z+10$. Solve a quadratic equation

$z^2-6z+10=0$

$D = (-6)^2 - 4 \cdot 10 = 36 - 40 = -4$

$$z_{1,2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

So, $z^2 - 6z + 10 = (z-3-i)(z-3+i)$, and $f(z) = \frac{z^2}{(z^2-6z+10)^2} = \frac{z^2}{(z-3-i)^2(z-3+i)^2}$

Use the theorem: $\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum_{z_k \in \mathbb{C}_+} Res(f(z), z_k)$

$f(z)$ has 2 singularities - $z_{1,2} = 3 \pm i$. Only $z_1 = 3 + i \in \mathbb{C}_+$ $(Im\, z_1 = 1 > 0)$ thus
$\int_{-\infty}^{\infty} \frac{z^2}{(z^2-6z+10)^2} dz = 2\pi i \cdot Res(\frac{z^2}{(z-3-i)^2(z-3+i)^2}, 3+i)$

If c is a pole of order n, then the residue of f around z = c can be found by the formula:

$$Res(f, c) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}} ((z-c)^n f(z))$$

$z = 3 + i$ is a pole of order 2, so

$$Res\left(\frac{z^2}{(z-3-i)^2(z-3+i)^2}, 3+i\right)$$
$$= \frac{1}{1!} \lim_{z \to 3+i} \frac{d}{dz} ((z-3-i)^2 \frac{z^2}{(z-3-i)^2(z-3+i)^2})$$
$$= \lim_{z \to 3+i} \frac{d}{dz} (\frac{z^2}{(z-3+i)^2}) = \lim_{z \to 3+i} \frac{2z(z-3+i)^2 - 2(z-3+i)z^2}{(z-3+i)^4}$$
$$= \frac{2(3+i)(3+i-3+i)^2 - 2(3+i-3+i)(3+i)^2}{(3+i-3+i)^4}$$
$$= \frac{2(3+i)(2i)^2 - 2(2i)(3+i)^2}{(2i)^4} = \frac{2(3+i)(-4) - 2(2i)(9+6i-1)}{16}$$
$$= \frac{-24 - 8i - 32i + 24}{16} = -\frac{40i}{16} = -\frac{5i}{2}$$

At last, $\int_{-\infty}^{\infty} \frac{z^2}{(z^2-6z+10)^2} dz = 2\pi i \cdot \left(-\frac{5i}{2}\right) = -5\pi i^2 = 5\pi$

Answer: 5π.

c) $\int_{-\infty}^{\infty} \frac{x \cos 3x}{x^2+2x+5} dx$

Solution:

We know the theorem, if $|f(z)|_{R=|z|} \leq \mu(R), \mu(R \to \infty) \to 0$, then

$$\int_{-\infty}^{\infty} f(x) e^{iax} dx = 2\pi i \sum_{z_k \in \mathbb{C}_+} Res(f(z) e^{iaz}, z_k), a > 0$$

$\int_{-\infty}^{\infty} \frac{x\cos 3x}{x^2+2x+5}\,dx = Re\left[\int_{-\infty}^{\infty} \frac{xe^{i3x}}{x^2+2x+5}\,dx\right]$, Re is a real part of complex number.

Do polynomial factorization of a denominator z^2+2z+5. Solve a quadratic equation

$z^2+2z+5=0$

$D=2^2-4\cdot 5=-16$

$$z_{1,2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

So, $z^2+2z+5=(z+1+2i)(z+1-2i)$. Use the theorem

$$\int_{-\infty}^{\infty} \frac{xe^{i3x}}{x^2+2x+5}\,dx = 2\pi i \sum_{z_k\in\mathbb{C}_+} Res\left(\frac{ze^{i3z}}{z^2+2z+5}, z_k\right)$$

Only $z_1=-1+2i\in\mathbb{C}_+$, that's why

$$2\pi i \sum_{z_k\in\mathbb{C}_+} Res\left(\frac{ze^{i3z}}{z^2+2z+5}, z_k\right) = 2\pi i\cdot Res\left(\frac{ze^{i3z}}{z^2+2z+5}, -1+2i\right)$$

At a simple pole c, the residue of f is given by:

$$Res(f,c)=\lim_{z\to c}(z-c)f(z)$$

$z_1=-1+2i$ is a simple pole, so

$$Res\left(\frac{ze^{i3z}}{z^2+2z+5}, -1+2i\right) = \lim_{z\to -1+2i}(z+1-2i)\frac{ze^{i3z}}{(z+1+2i)(z+1-2i)} = \frac{(-1+2i)e^{i3(-1+2i)}}{(-1+2i+1+2i)} = \frac{(-1+2i)e^{(-3i)}e^{-6}}{4i}$$ and

$$2\pi i\cdot Res\left(\frac{ze^{i3z}}{z^2+2z+5}, -1+2i\right) = 2\pi i\cdot\frac{(-1+2i)e^{(-3i)}e^{-6}}{4i}$$
$$=\frac{\pi}{2}e^{(-6)}(-1+2i)(\cos 3 - i\sin 3)$$
$$=\frac{\pi}{2}e^{(-6)}(-\cos 3+2\sin 3+2i\cos 3+i\sin 3)$$

$$\int_{-\infty}^{\infty} \frac{x\cos 3x}{x^2+2x+5}\,dx = Re\left[\frac{\pi}{2}e^{(-6)}(-\cos 3+2\sin 3+2i\cos 3+i\sin 3)\right]$$
$$=\frac{\pi}{2}e^{(-6)}(-\cos 3+2\sin 3)$$

Answer: $\frac{\pi}{2e^6}(-\cos 3+2\sin 3)$

3. Let function f(z) be analytic on 0<|z-z_0 |<R. Prove that integrals of f are path independent on 0<|z-z_0 |<R if and only if Resz0f=0

Solution:

The residue is a complex number proportional to the contour integral of a meromorphic function along a path enclosing one of its singularities. Analyticity of the Cauchy means that f(z) is analytical if for any close curve $\Gamma \subset A \int_{\Gamma} f(z)dz = 0$. If Resz0f(z)=0 that $\oint_{\gamma} f(z)dz = 2\pi i Res(f(z), z_0) = 0$. So f(z) has integral properties of analytical function and integrals of f(z) are path independent on 0<|z-z_0|<R.