Sample: Real Analysis - Mathematical Modelling

Q2. We are to minimize the following:

$$F(C_A, C_B) = \sum_{i=1}^{4} (C_A e^{-0.03t_i} + C_B e^{-0.05t_i} - y_i)^2$$

Use derivatives to find a point of minimum of the function of two variables. Derive all the derivatives:

$$F'_{C_A}(C_A, C_B) = \left(\sum_{i=1}^{4} (C_A e^{-0.03t_i} + C_B e^{-0.05t_i} - y_i)^2\right)'_{C_A}$$

$$= \sum_{i=1}^{4} 2(C_A e^{-0.03t_i} + C_B e^{-0.05t_i} - y_i) e^{-0.03t_i}$$

$$= \sum_{i=1}^{4} 2(C_A e^{-0.06t_i} + C_B e^{-0.08t_i} - y_i e^{-0.03t_i})$$

$$= 2\left(C_A \sum_{i=1}^{4} e^{-0.06t_i} + C_B \sum_{i=1}^{4} e^{-0.08t_i} - \sum_{i=1}^{4} y_i e^{-0.03t_i}\right)$$

$$F'_{C_B}(C_A, C_B) = \left(\sum_{i=1}^{4} (C_A e^{-0.03t_i} + C_B e^{-0.05t_i} - y_i)^2\right)'_{C_B}$$

$$= \sum_{i=1}^{4} 2(C_A e^{-0.03t_i} + C_B e^{-0.05t_i} - y_i) e^{-0.05t_i}$$

$$= \sum_{i=1}^{4} 2(C_A e^{-0.08t_i} + C_B e^{-0.1t_i} - y_i e^{-0.05t_i})$$

$$= 2\left(C_A \sum_{i=1}^{4} e^{-0.08t_i} + C_B \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i}\right)$$

$$F''_{C_A C_A}(C_A, C_B) = \left(2\left(C_A \sum_{i=1}^{4} e^{-0.06t_i} + C_B \sum_{i=1}^{4} e^{-0.08t_i} - \sum_{i=1}^{4} y_i e^{-0.03t_i}\right)\right)'_{C_A} = 2\sum_{i=1}^{4} e^{-0.06t_i}$$

$$F''_{C_B C_B}(C_A, C_B) = \left(2\left(C_A \sum_{i=1}^{4} e^{-0.08t_i} + C_B \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i}\right)\right)'_{C_B} = 2\sum_{i=1}^{4} e^{-0.1t_i}$$

$$F''_{C_A C_B}(C_A, C_B) = \left(2\left(C_A \sum_{i=1}^{4} e^{-0.06t_i} + C_B \sum_{i=1}^{4} e^{-0.08t_i} - \sum_{i=1}^{4} y_i e^{-0.03t_i}\right)\right)'_{C_B} = 2\sum_{i=1}^{4} e^{-0.08t_i}$$

Now, use Excel to calculate the sums required. We get:

$\sum_{i=1}^{4} e^{-0.06t_i}$	$\sum_{i=1}^{4} e^{-0.08t_i}$	$\sum_{i=1}^{4} e^{-0.1t_i}$	$\sum_{i=1}^{4} y_i e^{-0.03t_i}$	$\sum_{i=1}^{4} y_i e^{-0.05t_i}$
2.714325	2.387605	2.101257	27.6187	24.29845

Use the first derivative to find the critical point. Solve the following system of equations:

$$\begin{cases} F'_{C_A}(C_A, C_B) = 0 \\ F'_{C_B}(C_A, C_B) = 0 \end{cases} => \begin{matrix} C_A \sum_{i=1}^{4} e^{-0.06t_i} + C_B \sum_{i=1}^{4} e^{-0.08t_i} - \sum_{i=1}^{4} y_i e^{-0.03t_i} = 0 \\ C_A \sum_{i=1}^{4} e^{-0.08t_i} + C_B \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i} = 0 \end{matrix} =>$$

$$\begin{matrix} C_A \sum_{i=1}^{4} e^{-0.06t_i} \sum_{i=1}^{4} e^{-0.1t_i} + C_B \sum_{i=1}^{4} e^{-0.08t_i} \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.03t_i} \sum_{i=1}^{4} e^{-0.1t_i} = 0 \\ C_A \sum_{i=1}^{4} e^{-0.08t_i} \sum_{i=1}^{4} e^{-0.08t_i} + C_B \sum_{i=1}^{4} e^{-0.1t_i} \sum_{i=1}^{4} e^{-0.08t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i} \sum_{i=1}^{4} e^{-0.08t_i} = 0 \end{matrix} =>$$

$$C_A \left(\sum_{i=1}^{4} e^{-0.06t_i} \sum_{i=1}^{4} e^{-0.1t_i} - \left(\sum_{i=1}^{4} e^{-0.08t_i} \right)^2 \right) - \sum_{i=1}^{4} y_i e^{-0.03t_i} \sum_{i=1}^{4} e^{-0.1t_i} + \sum_{i=1}^{4} y_i e^{-0.05t_i} \sum_{i=1}^{4} e^{-0.08t_i} = 0$$

$$C_A \sum_{i=1}^{4} e^{-0.08t_i} + C_B \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i} = 0$$

$$=> \begin{matrix} C_A = \dfrac{\sum_{i=1}^{4} y_i e^{-0.03t_i} \sum_{i=1}^{4} e^{-0.1t_i} - \sum_{i=1}^{4} y_i e^{-0.05t_i} \sum_{i=1}^{4} e^{-0.08t_i}}{\sum_{i=1}^{4} e^{-0.06t_i} \sum_{i=1}^{4} e^{-0.1t_i} - (\sum_{i=1}^{4} e^{-0.08t_i})^2} \\ C_B = \dfrac{\sum_{i=1}^{4} y_i e^{-0.05t_i} - C_A \sum_{i=1}^{4} e^{-0.08t_i}}{\sum_{i=1}^{4} e^{-0.1t_i}} \end{matrix}$$

Use Excel to calculate the numerical values of the parameters:

$$\begin{cases} C_A \approx 6.657 \\ C_B \approx 3.999 \end{cases}$$

Now use the second order derivatives to check if the critical point is a point of minimum.

$$D(C_A, C_B) = F''_{C_B C_B}(C_A, C_B) * F''_{C_A C_A}(C_A, C_B) - \left(F''_{C_B C_B}(C_A, C_B) \right)^2$$

$$= 2 \sum_{i=1}^{4} e^{-0.06t_i} * 2 \sum_{i=1}^{4} e^{-0.1t_i} - 2 \sum_{i=1}^{4} e^{-0.08t_i} * 2 \sum_{i=1}^{4} e^{-0.08t_i} \approx 0.011 > 0$$

$$F''_{C_B C_B}(C_A, C_B) = 2 \sum_{i=1}^{4} e^{-0.08t_i} \approx 2.39 > 0$$

Both D and $F''_{C_B C_B}(C_A, C_B)$ are positive, thus the point $(C_A, C_B) \approx (6.657, 3.999)$ is the point of minimum of the function $F(C_A, C_B)$. Thus, these values of the parameters fit the model (1) in the best way.