Sample: Quantum Mechanics - Final Homework

Problem 1

a) Let us use recurrence relation $sH_n=\frac{1}{2}H_{n+1}+nH_{n-1}$ (we write H_n instead of $H_n(s)$ in order to shorten the equations). In order to obtain s^4H_n , one has to multiply sH_n by s three times, while using the recurrence relation after each multiplication.
Thus,

$$s^2H_n=\frac{s}{2}H_{n+1}+nsH_{n-1}=\frac{1}{2}(\frac{1}{2}H_{n+2}+(n+1)H_n)+n(\frac{1}{2}H_n+(n-1)H_{n-2})=\frac{1}{4}H_{n+2}+nH_n+\frac{1}{2}H_n+n(n-1)H_{n-2}$$

$$s^3H_n=s(s^2H_n)=\frac{1}{4}(\frac{1}{2}H_{n+3}+(n+2)H_{n+1})+n(\frac{1}{2}H_{n+1}+nH_{n-1})+\frac{1}{2}(\frac{1}{2}H_{n+1}+nH_{n-1})+n(n-1)(\frac{1}{2}H_{n-1}+(n-2)H_{n-3})=$$

$$=\frac{1}{8}H_{n+3}+\frac{3}{4}(n+1)H_{n+1}+\frac{3}{2}n^2H_{n-1}+n(n-1)(n-2)H_{n-3} \quad .$$

$$s^4H_n=s(s^3H_n)=\frac{1}{8}(\frac{1}{2}H_{n+4}+(n+3)H_{n+2})+\frac{3}{4}(n+1)(\frac{1}{2}H_{n-2}+(n+1)H_n)+\frac{3}{2}n^2(\frac{1}{2}H_n+(n-1)H_{n-2})+$$

$$+n(n-1)(n-2)(\frac{1}{2}H_{n-2}+(n-3)H_{n-4})=$$

$$\frac{1}{16}H_{n+4}+H_{n+2}(\frac{n+3}{8}+\frac{3}{8}(n+1))+H_n(\frac{3}{4}(n+1)^2+\frac{3}{4}n^2)+H_{n-2}(\frac{3}{2}n^2(n-1)+\frac{n(n-1)(n-2)}{2})+$$

$$+n(n-1)(n-2)(n-3)H_{n-4}=$$

$$\frac{1}{16}H_{n+4}+\frac{1}{(2n+3)}H_{n+2}+\frac{3}{4}[(n+1)^2+n^2]H_n+n(n-1)(2n-1)H_{n-2}+\frac{n!}{(n-4)!}H_{n-4} \quad .$$

b)

$$s^4\psi_n^0=\frac{e^{\frac{-s^2}{2}}}{\sqrt{a\sqrt{\pi}2^n n!}}\left(\frac{1}{16}H_{n+4}+\frac{1}{(2n+3)}H_{n+2}+\frac{3}{4}[(n+1)^2+n^2]H_n+n(n-1)(2n-1)H_{n-2}+\frac{n!}{(n-4)!}H_{n-4}\right)=$$

$$=\frac{e^{\frac{-s^2}{2}}}{\sqrt{a\sqrt{\pi}}}\left(\frac{H_{n+4}}{\sqrt{2^{n+4}(n+4)!}}\frac{1}{4}\sqrt{\frac{(n+4)!}{n!}}+\frac{H_{n+2}}{\sqrt{2^{n+2}(n+2)!}}\sqrt{\frac{(n+2)!}{n!}}(n+\frac{3}{2})+\frac{3}{4}[(n+1)^2+n^2]\frac{H_n}{\sqrt{2^n n!}}\right)+$$

$$+\frac{e^{\frac{-s^2}{2}}}{\sqrt{a\sqrt{\pi}}}\left(\frac{H_{n-2}}{\sqrt{2^{n-2}(n-2)!}}\sqrt{\frac{n!}{(n-2)!}}(n-\frac{1}{2})+\frac{1}{4}\frac{H_{n-4}}{\sqrt{2^{n-4}(n-4)!}}\sqrt{\frac{n!}{(n-4)!}}\right)=$$

$$\frac{1}{4}\sqrt{\frac{(n+4)!}{n!}}\psi_{n+4}^0+(n+\frac{3}{2})\sqrt{\frac{(n+2)!}{n!}}\psi_{n+2}^0+\frac{3}{4}[(n+1)^2+n^2]\psi_n^0+(n-\frac{1}{2})\sqrt{\frac{n!}{(n-2)!}}\psi_{n-2}^0+\frac{1}{4}\sqrt{\frac{n!}{(n-4)!}}\psi_{n-4}^0$$

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c) According to perturbation theory, $E_n^1=\langle\psi_n^0|V|\psi_n^0\rangle=\epsilon\hbar\omega\langle\psi_n^0|s^4|\psi_n^0\rangle$. Using part b) and the fact that wave-functions are orthonormal, obtain

$$E_n^1=\epsilon\hbar\omega\langle\psi_n^0|s^4|\psi_n^0\rangle=\epsilon\hbar\omega\cdot\frac{3}{4}[(n+1)^2+n^2]=\frac{3}{4}\epsilon\hbar\omega[(n+1)^2+n^2] \quad .$$

d) Using orthonormal property of wave-functions $\langle\psi_m^0|\psi_n^0\rangle=\delta_{nm}$ and result from b), obtain:

$$C_{nm}^1=\langle\psi_m^0|s^4|\psi_n^0\rangle=$$

$$=\frac{1}{4}\sqrt{\frac{(n+4)!}{n!}}\delta_{n+4;m}+(n+\frac{3}{2})\sqrt{\frac{(n+2)!}{n!}}\delta_{n+2;m}+\frac{3}{4}[(n+1)^2+n^2]\delta_{nm}+(n-\frac{1}{2})\sqrt{\frac{n!}{(n-2)!}}\delta_{n-2;m}+\frac{1}{4}\sqrt{\frac{n!}{(n-4)!}}\delta_{n-4;m}$$

Problem 2

a) Let us rewrite matrix elements in different form:
$W_{\alpha\beta}=\int_0^\infty r\,dr\int_0^{2\pi}d\theta\,\bar{\psi}_\alpha V_1\psi_\beta=a^2\int_0^\infty \rho\,d\rho\int_0^{2\pi}d\theta\,\bar{\psi}_\alpha V_1\psi_\beta$. In polar coordinates, the perturbation is $V_1(\rho,\theta)=\epsilon\hbar\frac{\omega}{a^2}x(r,\theta)y(r,\theta)=\epsilon\hbar\frac{\omega}{a^2}r^2\sin\theta\cos\theta=\epsilon\hbar\frac{\omega}{2}\rho^2\sin 2\theta$.

It is obvious, that $\int_0^{2\pi}\sin 2\theta\,d\theta=0$, thus diagonal elements W_{00},W_{++},W_{--} , angular part of which contains this integral are equal to zero. Since $W_{ab}=\bar{W}_{ba}$, one only has to calculate W_{0+},W_{0-},W_{+-} coefficients.

Let us use integral $\int_0^\infty x^n e^{-x^2}dx=\frac{1}{2}\Gamma(\frac{1+n}{2})$ expressed using Gamma function and use following notation $A=\epsilon\hbar\frac{\omega}{2}$, so $V_1=A\rho^2\sin 2\theta$.

$$W_{0+}=A\frac{a^2}{a^2\sqrt{2\pi}}\int_0^\infty\rho^5(1-\rho^2)e^{-\rho^2}d\rho\int_0^{2\pi}e^{2i\theta}\sin 2\theta\,d\theta=\frac{A}{\sqrt{2}}\cdot\frac{1}{2}(\Gamma(3)-\Gamma(4))\cdot(i\pi)=\frac{A}{\sqrt{2}}\cdot\frac{1}{2}(2!-3!)(i\pi)=-\sqrt{2}iA$$
$$W_{0-}=A\frac{a^2}{a^2\sqrt{2\pi}}\int_0^\infty\rho^5(1-\rho^2)e^{-\rho^2}d\rho\int_0^{2\pi}e^{-2i\theta}\sin 2\theta\,d\theta=\frac{A}{\sqrt{2}}\cdot\frac{1}{2}(\Gamma(3)-\Gamma(4))\cdot(-i\pi)=\sqrt{2}Ai$$

The angular part of W_{+-} is $\int_0^{2\pi}e^{-4i\theta}e^{-\rho^2}\sin 2\theta\,d\theta$ and is equal to zero, so W_{+-} is equal to zero too. All matrix elements are found, and the matrix looks like this:

$$W=\begin{pmatrix}0 & -\sqrt{2}Ai & \sqrt{2}Ai\\ \sqrt{2}Ai & 0 & 0\\ -\sqrt{2}Ai & 0 & 0\end{pmatrix}\text{, where } A=\frac{\epsilon\hbar\omega}{2}\ .$$

b) The characteristic equation $det(W-E^1 I)=0$ is $det\begin{pmatrix}-E^1 & -\sqrt{2}Ai & \sqrt{2}Ai\\ \sqrt{2}Ai & -E^1 & 0\\ -\sqrt{2}Ai & 0 & -E^1\end{pmatrix}=0$. Opening the determinant, obtain $-(E^1)^3+4E^1A^2=0$, so the roots are $(E^1)_1=0$ and

$(E^1)_{2,3}=\mp 2A=\mp\epsilon\hbar\omega$.