Sample: Real Analysis - Written Optimization Analysis Problem

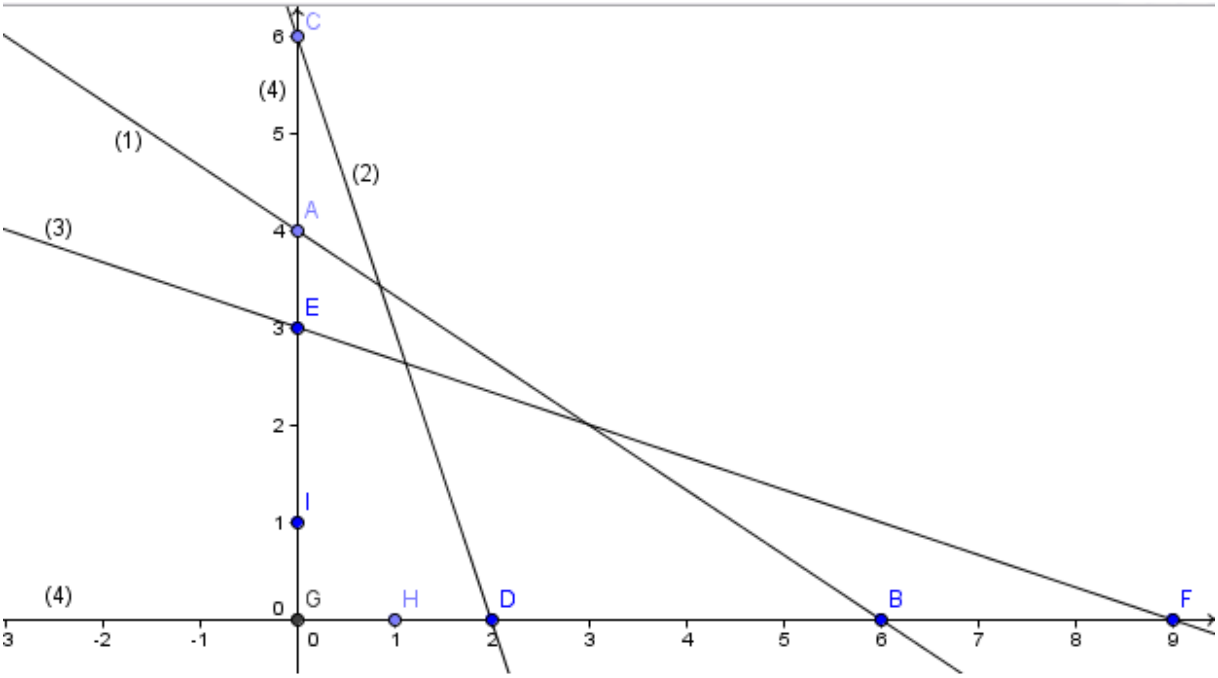
Problem 1
A common type of optimization problem is designing a diet that meets certain dietary requirements while maximizing certain values (protein, fiber, etc.) or minimizing other values (cost, fat content, cholesterol, etc.).
Rice and Soybeans are some of the most common staples in the world. Design the lowest cost diet that meets all the requirements for Protein, Calories, and Vitamin B2.
We want to insure that people on this diet receive at least 90g of protein, 1620 calories, 1 milligram of Vitamin B2.
An uncooked cup of rice contains 15 grams of protein, 810 calories, and 1/9 milligram of B2, all at a price of 21 cents.
An uncooked cup of soybeans costs 14 cents and contains 22.5 grams of protein, 270 calories, and 1/3 milligram of B2.
What is the cheapest diet that will fulfill the dietary requirements?
Solution
Since this is a two variable problem, we will use variables x and y to represent the values and solve with a grapher.
x : number of cups of rice per day
y : number of cups of soybeans per day

We have the following objective function:
$Cost = 21x + 14y$

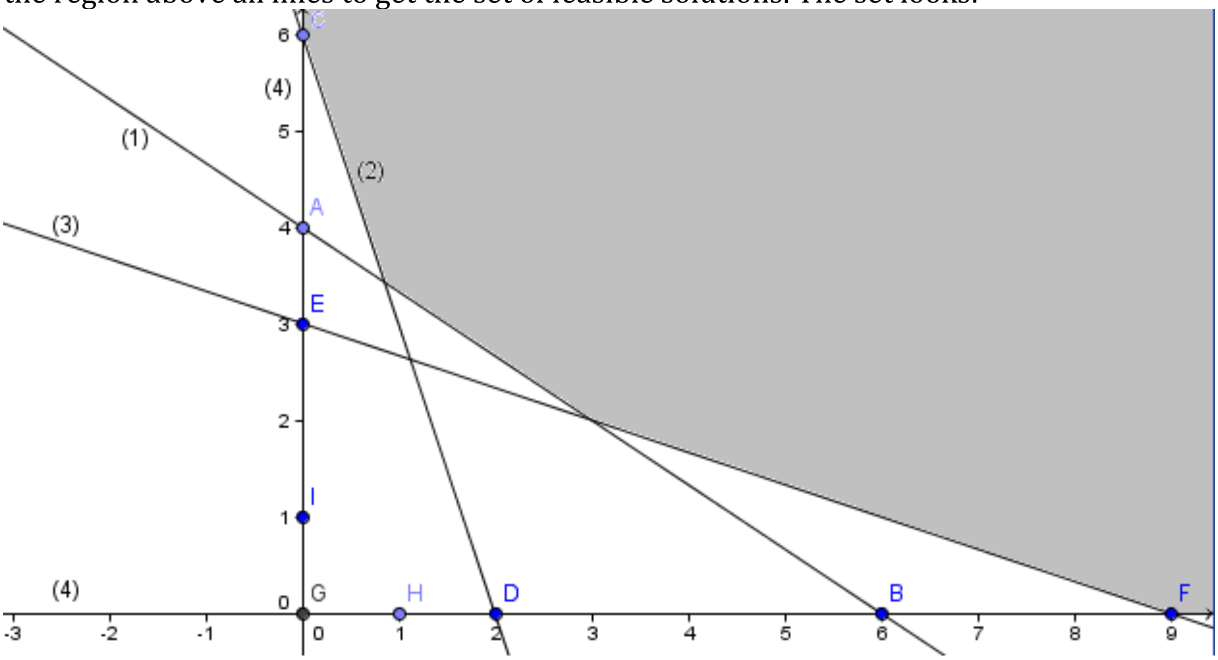
Subject to the following constraints:

$$\begin{cases} 15x + 22.5y \geq 90 \\ 810x + 270y \geq 1620 \\ \dfrac{1}{9}x + \dfrac{1}{3}y \geq 1 \\ x \geq 0, y \geq 0 \end{cases}$$

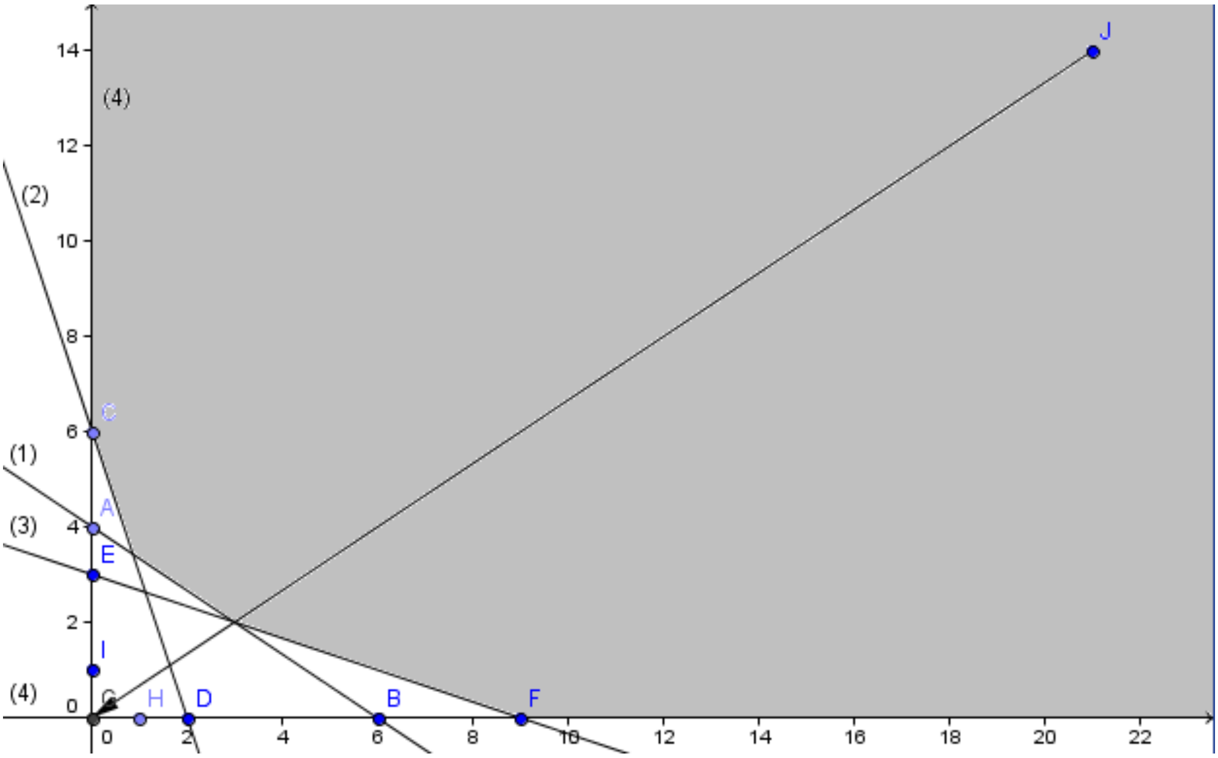
Build the region using GeoGebra. We can easily find 2 points that belong to each constraint:
For the firs: (0, 4), (6, 0)
For the second: (0, 6), (2, 0)
For the third: (0, 3), (9, 0)
For the fourth: there are 2 lines: the first contains (0, 0) and (0, 1) and the second – (0, 0) and (1, 0).
Use 'line through 2 points' to build graphs of constraints. The result looks (numbers in parentheses means number of constraint):



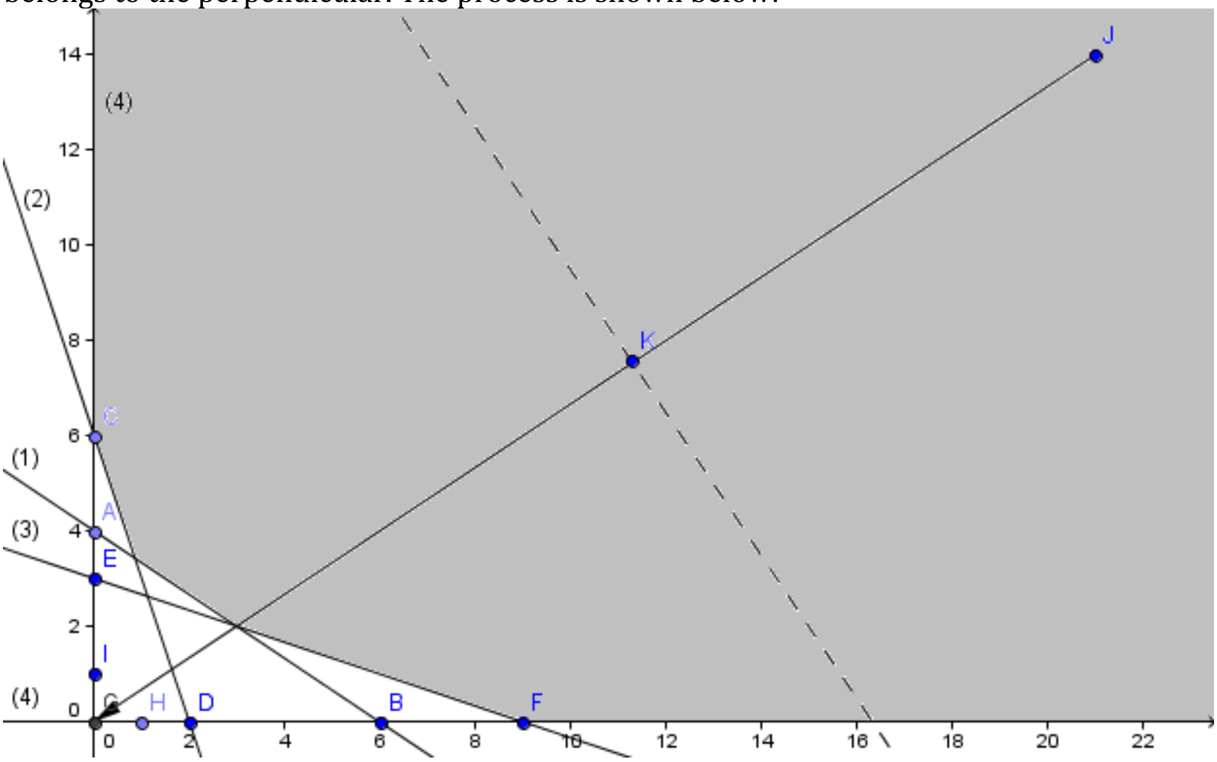
All constraints contain “>=” and “y” in the left part has positive sign, so we should fill the region above all lines to get the set of feasible solutions. The set looks:

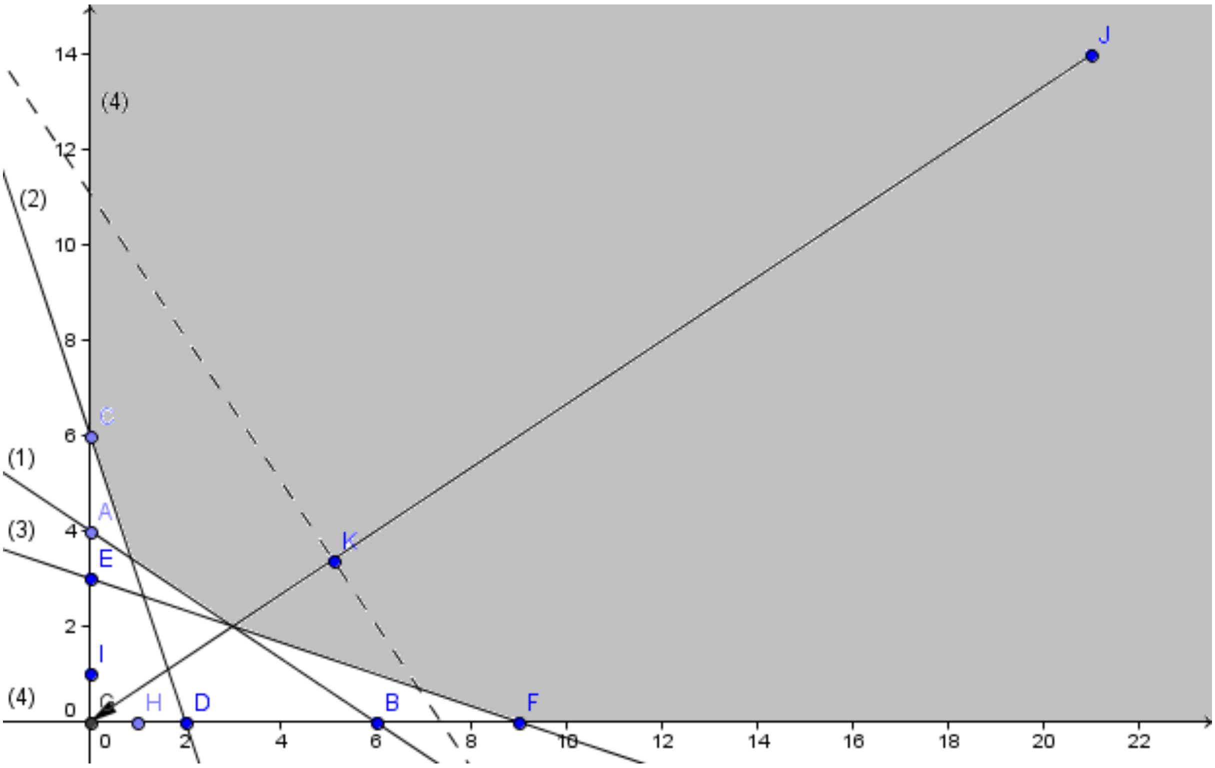


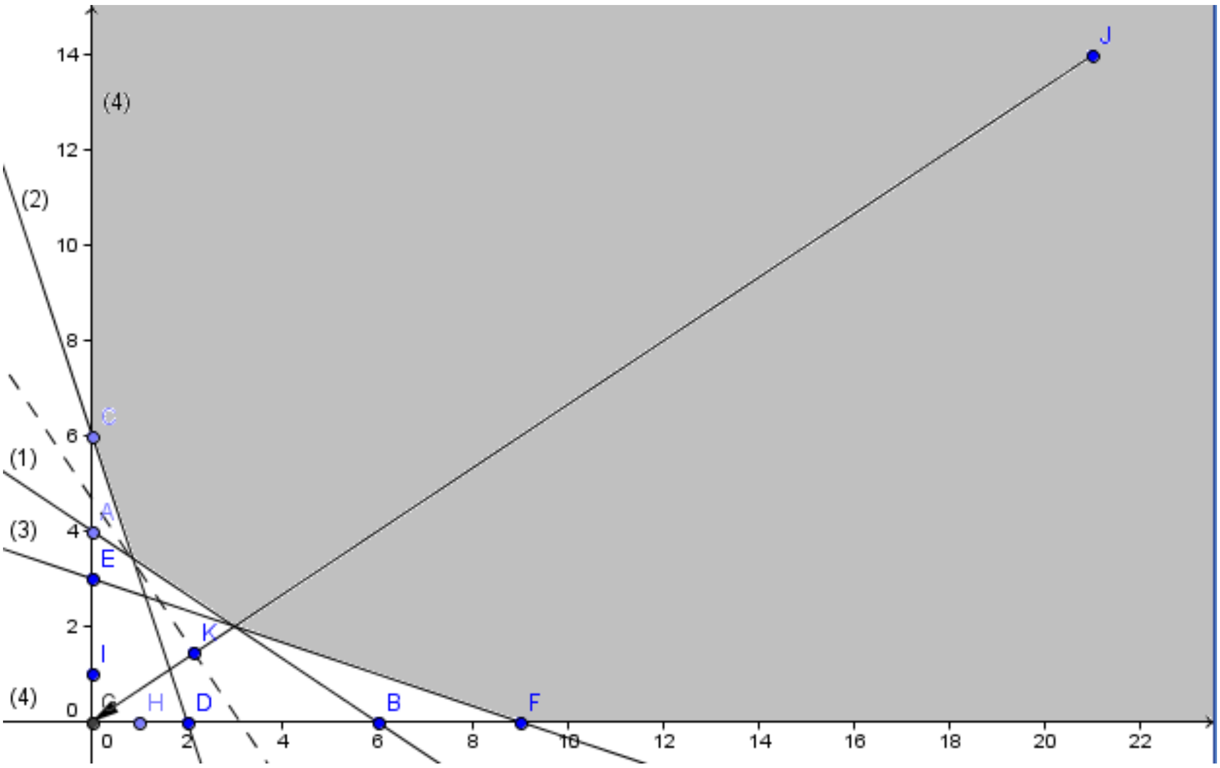
Draw the Objective vector now. It contains points (0, 0) and (21, 14) and tends to the origin:



The next step is: draw perpendicular to the Objective vector and move it from the first quartile to the origin until at least one point of the set of feasible solutions belongs to the perpendicular. The process is shown below:




J
(4)
(2)
C
(1)
A
(3)
K
E
I
(4)
G
H
D
B
F


J
(4)
(2)
C
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K
I
(4)
G
H
D
B
F

Only one point of the set of feasible solutions lies on the perpendicular now. So, it is the solution. This point is intersection of lines that corresponds to the constraints (2) and (1). Solve the system of correspondent equations:

$$\begin{cases} 15x + 22.5\,y = 90 \\ 810\,x + 270y = 1620 \end{cases} \leftrightarrow \begin{cases} 2x + 3\,y = 12 \\ 3\,x + y = 6 \end{cases} \leftrightarrow \begin{cases} x = 6/7 \\ y = 24/7 \end{cases}$$

The correspondent value of Objective function is:

P = 21*6/7 + 14*24/7 = 66

So, the cheapest diet that will fulfill the dietary requirements includes 6/7 cups of rice and 27/7 cups of Soybeans. Price of this diet is 66 cents.