Sample: Linear Algebra - Transformations

16)

(a)

Let f_1, f_2 be continuous real-valued functions on $[0,1]$. Then linear combination

$$c_1 f_1 + c_2 f_2$$

is also continuous real-valued function on $[0,1]$. So V is the linear subspace of $\mathfrak{F}([0,1])$.

(b)

Let $f_1, f_2 \in W$. Then

$$\int_0^1 f_1(t)dt = 0$$

$$\int_0^1 f_2(t)dt = 0$$

Then

$$\int_0^1 c_1 f_1(t) + c_2 f_2(t)dt = c_1 \int_0^1 f_1(t)dt + c_2 \int_0^1 f_2(t)dt = 0$$

Thus

$$c_1 f_1 + c_2 f_2 \in W$$

So W is a subspace of V.

17)

$f(x)$ is the polynomial of degree n. Thus $f'(x)$ is a polynomial f degree $n-1$, $f''(x)$ is a polynomial of degree $n-2$, …, $f^{(k)}(x)$ is a polynomial of degree $n-k$, $f^{(n)}(x)$ is a constant.. Since all polynomials $f, f', \ldots, f^{(n)}$ have different degree, they are linearly independent. Since they have consecutive degrees starting from 0,

$\{f, f', \dots, f^{(n)}\}$ form a basis of $P_n(\mathbb{R})$. So for every $g(x) \in P_n(\mathbb{R})$ exists $c_1, \dots, c_n$ such that

$$g(x) = c_1 f(x) + c_2 f'(x) + \dots + c_n f^{(n)}(x)$$

18)

a) Is not linear transformation

b) Is not linear transformation

c) Is not linear transformation

d) It is linear transformation. Matrix:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & \pi^2 \end{pmatrix}$$

e) Is not linear transformation

19)

Suppose vectors $x_1, \dots, x_r, v_1, \dots, v_s$ are not linearly independent. Then there exists their non-trivial linear combination that equals to 0:

$$c_1 x_1 + \dots + c_r x_r + d_1 v_1 + \dots + d_s v_s = 0$$

Since x_i are independent this linear combination contains at least one none-zero term d_i.

Let's apply operator T to this equality.

$$T(c_1 x_1 + \dots + c_r x_r + d_1 v_1 + \dots + d_s v_s) = T(0)$$

Since T is linear we have:

$$c_1 T(x_1) + \dots + c_r T(x_r) + d_1 T(v_1) + \dots + d_s T(v_s) = 0$$

Since $x_1, \dots, x_r \in N(T)$

$$T(x_i) = 0$$

So

$$d_1 T(v_1) + \cdots + d_s T(v_s) = 0$$

We got non-trivial linear combination of vectors $T(v_i)$ that equals to 0. This contradicts to independence of vectors $\{T(v_i)\}$.

So vectors $x_1, \dots, x_r, v_1, \dots, v_s$ are linearly independent.

20)

Let

$$T(v) = T\big((v_1, v_2, v_3)\big) = (0, v_1, v_2)$$

T is linear transformation with matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then

$$T\left(T\left(T\big((v_1, v_2, v_3)\big)\right)\right) = T\left(T\big((0, v_1, v_2)\big)\right) = T\big((0,0,v_1)\big) = (0,0,0) = 0$$

So we have $T\left(T\big(T(v)\big)\right) = 0$ for all $v \in \mathbb{R}^3$. But

$$T\left(T\big((1,1,1)\big)\right) = T\big((0,1,1)\big) = (0,0,1) \neq 0$$

21)

$$A_{ij} = y_i x_j$$

Matrix A has rows that are multiples of $(x_1, \dots, x_n)$ with coefficients $y_1, \dots, y_m$. Thus rank of matrix A equals to 1. Nullity of L_A equals to $n-1$.

22)

Kernel of operator T is a plane. Equation of the kernel:

$$9x - 9y + z = 0$$

Let's take

$$T = \begin{pmatrix} 9 & -9 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Kernel of T equals to

$$span\left(\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix}\right\}\right)$$

23)

(a)

$$T(f) = (x+1)f' + f$$

Let's check whether operator is linear:

$$T(c_1 f_1 + c_2 f_2) = (x+1)(c_1 f_1 + c_2 f_2)' + (c_1 f_1 + c_2 f_2)$$
$$= c_1\big((x+1)f_1' + f_1\big) + c_2\big((x+1)f_2' + f_2\big) = c_1 T(f_1) + c_2 T(f_2)$$

So T is linear.

(b)

Let's find explicit action of T on polynomials:

$$T(c_0 + c_1 x) = (x+1)c_1 + c_0 + c_1 x = c_0 + c_1 + 2c_1 x$$

Matrix representation of T:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

24)

Matrix representation of T:

$$T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Nullity of the operator:

$$v_2 = v_3 = \cdots = v_n = 0$$

So

$$N(T) = span\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}\right\}\right)$$

$$T^k(v_1, \dots, v_n) = (v_{1+k}, v_{2+k}, \dots, v_n, 0,0, \dots ,0)$$

$$N(T^k) = span\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}\right\}\right)$$