Sample: Analytic Geometry - The Second Order Curves

MATH 142

PRECALCULUS II

1. Assume that the vertex of parabola at the origin. Find an equation of the parabola that satisfies the given conditions, and sketch the graph.

(a) the focus is the point (0, 3)

Solution:

Both the focus and the vertex of parabola lie on the y-axis, so the parabola is the vertical parabola and it's an equation is:

$$4p(y-k) = (x-h)^2$$

Where $(h, k)$is the coordinates of the vertex and p is the distance from the vertex to the focus.

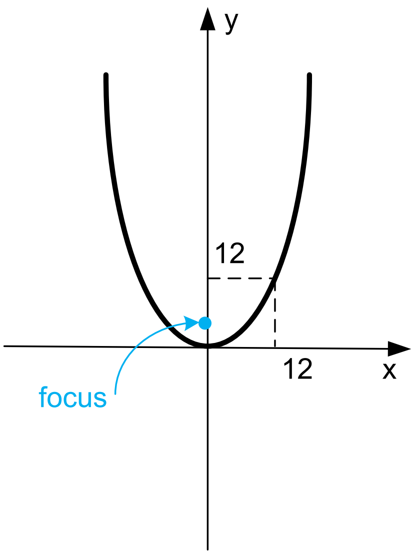
Given the coordinates of the vertex is $(0,0)$ and $p = 3$.

Thus,

$$4 \cdot 3(y-0) = (x-0)^2$$

$$12y = x^2$$

Answer: $\boldsymbol{y = \frac{1}{12}x^2}$



(b) the directrix is $y - 8 = 0$

Solution:

The directrix is a horizontal line, so the parabola is the vertical parabola and it's an equation is:

$4p(y-k) = (x-h)^2$ and an equation of the directrix is $y = k - p$

Where $(h, k)$is the coordinates of the vertex and p is the distance from the vertex to the focus.

Given the coordinates of the vertex is $(0,0)$and $y = 8$, so

$$h = 0, \qquad k = 0, \qquad k - p = 8$$

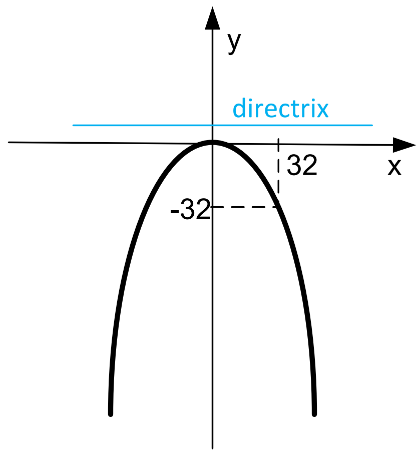
$$0 - p = 8, \qquad p = -8$$

Thus,

$$4 \cdot (-8)(y-0) = (x-0)^2$$

$$-32y = x^2$$

Answer: $\boldsymbol{y = -\frac{1}{32}x^2}$



(c) the focus is the smaller of the two x-intercepts of the circle $x^2 - 8x - 2 + y^2 - 6y + 9 = 0$

Solution:

Finding the x-intercepts of the circle $x^2 - 8x - 2 + y^2 - 6y + 9 = 0$

Substitute $y = 0$ into an equation of the circle:

$$x^2 - 8x - 2 + 9 = 0$$

$$x^2 - 8x + 7 = 0$$

Solve the equation:

$$x = \frac{8 \pm \sqrt{64 - 28}}{2} = \frac{8 \pm 6}{2}$$

The smaller of the two x-intercepts is $x = 1$, so the focus is the point (1,0)

Both the focus and the vertex of parabola lie on the x-axis, so the parabola is the horizontal parabola and it's an equation is:

$$4p(x - k) = (y - h)^2$$

Where $(h, k)$is the coordinates of the vertex and p is the distance from the vertex to the focus.

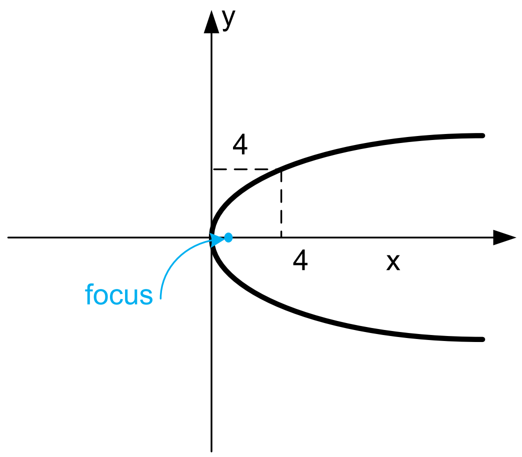
Given the coordinates of the vertex is $(0,0)$ and $p = 1$.

Thus,

$$4 \cdot 1(x - 0) = (y - 0)^2$$

$$4x = y^2$$

Answer: $\boldsymbol{x = \frac{1}{4}y^2}$



2. Find the distance between the vertices of the parabolas $y = -\frac{1}{2}x^2 + 4x$ and $y = 2x^2 - 8x - 1$

Solution:

Find the vertices of the parabolas.

Each parabola has a vertical line of symmetry that passes through its vertex. Because of this symmetry, the line of symmetry would, for example, pass through the midpoint of the two x-intercepts of a parabola. We can find the x-coordinate of that midpoint, and consequently, the x-coordinate of the vertex, by taking the average of the two solutions to the quadratic formula since this formula gives precisely the two x-intercepts of a parabola.

So for the first parabola:

$$-\frac{1}{2}x^2 + 4x = 0$$

$x = 0$ and $x = 8$, so the x-coordinate of the vertex is:

$$x = \frac{0+8}{2} = 4$$

When $x = 4$, then $y = -\frac{1}{2} \cdot 4^2 + 4 \cdot 4 = 8$

Thus, the vertex of the parabola is $(4{,}8)$

For the second parabola:

$$2x^2 - 8x - 1 = 0$$

$$x = \frac{8 \pm \sqrt{16+8}}{4} = \frac{8 \pm \sqrt{24}}{4}$$

$x = \frac{8-\sqrt{24}}{4}$ and $x = \frac{8+\sqrt{24}}{4}$, so the x-coordinate of the vertex is:

$$x = \frac{\frac{8-\sqrt{24}}{4} + \frac{8+\sqrt{24}}{4}}{2} = 2$$

When $x = 2$, then $y = 2 \cdot 2^2 - 8 \cdot 2 - 1 = -9$

Thus, the vertices of the parabola is $(2, -9)$

The distance between two points is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2-4)^2 + (-9-8)^2} = \sqrt{293}$$

Answer: the distance between the vertices is $\sqrt{293}$

3. Sketch the graph of $2y^2 - 3x + 28y + 110 = 0$

Solution:

$$2y^2 - 3x + 28y + 110 = 0$$

$$\frac{3}{2}x = y^2 + 14y + 55$$

This parabola is the horizontal parabola.

To find the vertices of the parabola, find its equation of the form:

$$4p(x-h)=(y-k)^2$$

$$4p=\frac{3}{2}$$

$$\frac{3}{2}x-\frac{3}{2}h=(y-k)^2$$

$\frac{3}{2}x=y^2-2ky+k^2+\frac{3}{2}h$, so

$$y^2-2ky+k^2+\frac{3}{2}h=y^2+14y+55$$

$$-2k=14$$

$$k^2+\frac{3}{2}h=55$$

Thus, $k=-7$

$$(-7)^2+\frac{3}{2}h=55$$

$$h=\frac{55-49}{\frac{3}{2}}=4$$

$(h,k)$is the coordinates of the vertex, so the vertex is $(4,-7)$

The line of symmetry would pass through the vertex parallel to x-axis: $y=-7$

There are no y- intercept, x- intercept is $\left(\frac{110}{3},0\right)$