Sample: Real Analysis - Real Analysis Task

Problem 1. Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers with the property

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L \in [-\infty, +\infty].$$

Prove that the sequences $a_n = \frac{x_n+y_n}{2}$ and $b_n = \sqrt{x_n y_n}$ are convergent to L.
(For $\{b_n\}$, it is assumed that $x_n, y_n \geq 0$, of course.)

Solution. Suppose $\varepsilon > 0$.

Since we are given $\lim_{n\to\infty} x_n = L$, then by definition of the limit of a sequence there exists some $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|x_n - L| < \frac{\varepsilon}{2}$.

Similarly, since $\lim_{n\to\infty} y_n = L$, there exists some $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|y_n - L| < \frac{\varepsilon}{2}$.

Let us now choose $N = \max\{N_1, N_2\}$. For $n > N$, both of the above inequalities hold. We can therefore add them together:

$$|x_n - L| + |y_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $n > N$.

Next, recalling the triangle inequality, we have

$$|(x_n + y_n) - (L + L)| \leq |x_n - L| + |y_n - L|,$$

and thus

$$|(x_n + y_n) - 2L| < \varepsilon.$$

We see that $\lim_{n\to\infty} (x_n + y_n) = 2L$.

Finally, recall the following property of limits of real sequences:

$$\lim_{n\to\infty} c\, x_n = c \lim_{n\to\infty} x_n \quad \text{for every } c \in \mathbb{R}.$$

By applying this property, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}(x_n + y_n) = \frac{1}{2} \lim_{n\to\infty} (x_n + y_n) = \frac{1}{2} * 2L = L,$$

and we have completed the first part of the proof.

Let us now look at sequence b_n.

This part will be slightly more complicated, since we will need to use an additional result: *every convergent sequence is bounded*. Let us prove this statement.

We will use $\{x_n\}$ as an example. Recall that we have $\lim_{n\to\infty} x_n = L \in \mathbb{R}$, which means that $\{x_n\}$ is a convergent sequence. According to the definition given above, there exists some $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|x_n - L| < \varepsilon$, or $L - \varepsilon < x_n < L + \varepsilon$.

The set $\{x_n\colon\ 1 \leq n \leq N_1\}$ is finite and therefore bounded: there exist $m, M \in \mathbb{R}$ such that for all $n \leq N_1$, we have $m < x_n < M$.

Now take $m' = \min\{L - \varepsilon,\ m\}$ and $M' = \max\{L + \varepsilon, M\}$. We now have that for all $n \in \mathbb{N}$, $m' < x_n < M'$, so the sequence $\{x_n\}$ is bounded.

We can now proceed to the final part of our proof.

Applying the result above to both $\{x_n\}$ and $\{y_n\}$, we can state that there exist $m', m'', M', M'' \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $m' < x_n < M'$ and $m'' < y_n < M''$. We now choose $M = \max\{1, |L|, |m'|, |m''|, |M'|, |M''|\}$.

We can repeat our definition of convergence for x_n and y_n:

- there exists some $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|x_n - L| < \frac{\varepsilon}{2M}$;
- there exists some $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|y_n - L| < \frac{\varepsilon}{2M}$.

Just like for a_n, choose $N = \max\{N_1, N_2\}$, and both inequalities hold for $n > N$.

Now let us evaluate $|x_n y_n - L^2|$.

$$|x_n y_n - L^2| = |x_n y_n - x_n L + x_n L - L^2| = |x_n(y_n - L) + L(x_n - L)|$$

Apply the triangle inequality:

$$|x_n(y_n - L) + L(x_n - L)| \le |x_n| * |y_n - L| + |L| * |x_n - L| \le M\frac{\varepsilon}{2M} + |L|\frac{\varepsilon}{2M}.$$

But $|L| \le M$ by definition of M, so

$$M\frac{\varepsilon}{2M} + |L|\frac{\varepsilon}{2M} \le 2M\frac{\varepsilon}{2M} = \varepsilon.$$

Combining the first and last steps, we have

$$|x_n y_n - L^2| \le \varepsilon,$$

which means that $\lim_{n\to\infty} x_n y_n = L^2$.

Suppose that $L \ne 0$. As we have just shown, there exists an $N \in \mathbb{N}$ such that for every $n > N$, $|x_n y_n - L^2| \le \varepsilon$, which can be modified to $|x_n y_n - L^2| \le \varepsilon|L|$.

Now consider

$$\left|\sqrt{x_n y_n} - L\right| = \left|\sqrt{x_n y_n} - L\right| \frac{\left|\sqrt{x_n y_n} + L\right|}{\left|\sqrt{x_n y_n} + L\right|} = \frac{|x_n y_n - L^2|}{\left|\sqrt{x_n y_n} + L\right|} \le \frac{|x_n y_n - L^2|}{|L|} < \frac{\varepsilon|L|}{|L|} = \varepsilon.$$

Thus, if $L \ne 0$, we have $\lim_{n\to\infty} \sqrt{x_n y_n} = L$.

Now suppose that $L = 0$. Since $\lim_{n\to\infty} x_n y_n = L^2 = 0$, there exists some $N \in \mathbb{N}$ such that for every $n > N$, $x_n y_n = |x_n y_n - 0| < \varepsilon^2$ (note that here we have used the condition that $x_n, y_n \ge 0$). Then $\sqrt{x_n y_n} < \sqrt{\varepsilon^2} = \varepsilon$, and $\left|\sqrt{x_n y_n} - L\right| = \left|\sqrt{x_n y_n}\right| < \varepsilon$, and in this case we also have $\lim_{n\to\infty} \sqrt{x_n y_n} = L$.

The proof is complete.

Problem 2. Find the radius of convergence of the following power-series

$$\sum_{n=0}^{\infty} \frac{1 + (-1)^n}{\sqrt{n}\, 2^n} z^n,$$

specify the disk of convergence, and study the convergence at the points z on the boundary of that disk situated on the real line, respectively on the y-axis.

(ATTN: We are taking about four values of z here.)

Solution. To find the radius of convergence, we will first need to find the limit superior of the sequence $\left\{(c_n)^{\frac{1}{n}}\right\}$, where (in our case) $c_n = \frac{1+(-1)^n}{\sqrt{n}\, 2^n}$. To do this, note that $c_{2k} = \frac{1+1}{\sqrt{2k}\, 2^{2k}} = \frac{2}{\sqrt{2k}\, 2^{2k}} > 0$, whereas $c_{2k+1} = \frac{1-1}{\sqrt{2k+1}\, 2^{2k+1}} = 0$. Thus,

$$\lim_{n\to\infty} \sup (c_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sup \left(\frac{1 + (-1)^n}{\sqrt{n}\, 2^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2}{\sqrt{n}\, 2^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2^{\frac{1}{n}}}{2\left(\sqrt{n}\right)^{\frac{1}{n}}}.$$

Now note that $\lim_{n\to\infty} n^{\frac{1}{2n}} = \lim_{n\to\infty} e^{\frac{1}{2n}\ln n} = 1$ and $\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1$. Therefore,

$$\lim_{n\to\infty} \sup (c_n)^{\frac{1}{n}} = \frac{1}{2 * 1} = \frac{1}{2}.$$

Applying the formula for **radius of convergence**, we have

$$R = \frac{1}{\lim_{n\to\infty} \sup (c_n)^{\frac{1}{n}}} = 2.$$

The **disk of convergence** is $\Delta_R = \{z : |z| < 2\}$.

The last part of the problem is to study the convergence at the points z on the boundary of that disk situated on the real line, respectively on the y-axis. These points are $(2,0), (0,2), (-2,0), (0,-2)$.

- $z_1 = 2$

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}2^n=\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

Note that the series $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ diverges, since we know that power series $\sum_{n=0}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$ and in our case $p=\frac{1}{2}<1$.

However, $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges, which can be shown by the alternating series test, since the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$ decreases monotonically and goes to zero in the limit as $n\to\infty$.

Now recall that the sum of ta convergent and divergent series diverges. Thus, $\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}2^n$ **diverges**.

- $z_2=2i$

Recall that $i^{4k+1}=i,\ i^{4k+2}=-1,\ i^{4k+3}=-i,\ i^{4k}=1$ for every $k\in\mathbb{N}$. Therefore,

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}(2i)^n=$$

$$=\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+1}}{\sqrt{4n+1}}i-\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+2}}{\sqrt{4n+2}}-\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+3}}{\sqrt{4n+3}}i+\sum_{n=0}^{\infty}\frac{1+(-1)^{4n}}{\sqrt{4n}}=i\sum_{n=0}^{\infty}\left(\frac{1+(-1)^{4n+1}}{\sqrt{4n+1}}-\frac{1+(-1)^{4n+3}}{\sqrt{4n+3}}\right)+\sum_{n=0}^{\infty}\left(\frac{1+(-1)^{4n}}{\sqrt{4n}}-\frac{1+(-1)^{4n+2}}{\sqrt{4n+2}}\right)$$

Now note that $(-1)^{4n}=(-1)^{4n+2}=1$ and $(-1)^{4n+1}=(-1)^{4n+3}=-1$. Thus, swe can further simplify this expression as follows:

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}(2i)^n=i\sum_{n=0}^{\infty}\left(\frac{1-1}{\sqrt{4n+1}}-\frac{1-1}{\sqrt{4n+3}}\right)+\sum_{n=0}^{\infty}\left(\frac{1+1}{\sqrt{4n}}-\frac{1+1}{\sqrt{4n+2}}\right)$$

$$=\sum_{n=0}^{\infty}\left(\frac{2}{\sqrt{4n}}-\frac{2}{\sqrt{4n+2}}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{2}{\sqrt{4n+2}}\right)$$

Let us now investigate convergence of this series. We will transform the summand:

$$\frac{1}{\sqrt{n}}-\frac{2}{\sqrt{4n+2}}=\frac{\sqrt{4n+2}-2\sqrt{n}}{\sqrt{n}\sqrt{4n+2}}=\frac{4n+2-4n}{\sqrt{n}\sqrt{4n+2}\left(\sqrt{4n+2}+2\sqrt{n}\right)}=$$

$$=\frac{2}{\sqrt{n}\sqrt{4n+2}\left(\sqrt{4n+2}+2\sqrt{n}\right)}$$

We will use the comparison convergence test. Since $\sqrt{4n+2}>\sqrt{4n}$, we have

$$\frac{2}{\sqrt{n}\sqrt{4n+2}\left(\sqrt{4n+2}+2\sqrt{n}\right)}\le\frac{2}{\sqrt{n}\sqrt{4n}\left(\sqrt{4n}+2\sqrt{n}\right)}=\frac{2}{2n\left(2\sqrt{n}+2\sqrt{n}\right)}=\frac{1}{3n\sqrt{n}}=$$

$$=\frac{1}{3n^{\frac{3}{2}}}$$

Recall again that the power series $\sum_{n=0}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$; here $p=\frac{3}{2}>1$, so our series $\sum_{n=0}^{\infty}\frac{1}{3n^{\frac{3}{2}}}$ converges.

Finally, by applying the comparison convergence test, we see that the initial series $\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}(2i)^n$ also **converges**.

- $z_3=-2$

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}(-2)^n=\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}}(-1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}+\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$$

This expression is equal to the one we obtained for $\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}2^n$, and we have already shown that this series **diverges** above.

- $z_4=-2i$

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}\,2^n}(-2i)^n=\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}}(-i)^n$$

We will use the same approach as for $z_2=2i$.

$$\sum_{n=0}^{\infty}\frac{1+(-1)^n}{\sqrt{n}}(-i)^n=$$

$$=-\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+1}}{\sqrt{4n+1}}i-\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+2}}{\sqrt{4n+2}}+\sum_{n=0}^{\infty}\frac{1+(-1)^{4n+3}}{\sqrt{4n+3}}i+\sum_{n=0}^{\infty}\frac{1+(-1)^{4n}}{\sqrt{4n}}=$$

Problem 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with the property $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to-\infty}f(x)=+\infty$. Prove that such a function attains its minimum.

Solution. Let us first consider what we mean by $\lim\limits_{x\to+\infty}f(x)=+\infty$: for every $M>0$, there exists some $n_1>0$ such that for all $x>n_1$, we have $f(x)>M$.

Similarly, $\lim\limits_{x\to-\infty}f(x)=+\infty$ is equivalent to the statement that for every $M>0$, there exists some $n_2>0$ such that for all $x<-n_2$, $f(x)>M$.

Therefore, for every $M>0$ we can choose $n=\max\{n_1,n_2\}$ so that if $|x|>n$, then $f(x)>M$. We see that f does not attain its minimum outside $[-n,n]$.

But $[-n,n]$ is a compact set. Since the function f is continuous, it attains a minimum on $[-n,n]$ (by the Extreme Value Theorem). Let us denote the point where the minimum is attained as x_0: $f(x_0)=\min\limits_{x\in[-n,n]}f(x)$.

Due to the way we chose n, $f(x)<M$ for all $x\in[-n,n]$; thus, $f(x_0)<M$, and we see that $f(x_0)=\min\limits_{x\in\mathbb{R}}f(x)$. The proof is complete.

Problem 4. Given that $f:\mathbb{R}\to\mathbb{R}$ is differentiable at 0 and $f'(0)=1$, find

$$\lim_{x\to0}\frac{f(x)-f(-x)}{x}.$$

Give reasons for your answer.

Solution. To find the value of our expression, we will somewhat transform it by adding and subtracting $f(0)$:

$$\lim_{x\to0}\frac{f(x)-f(-x)}{x}=\lim_{x\to0}\frac{f(x)-f(0)+f(0)-f(-x)}{x}=\lim_{x\to0}\frac{f(x)-f(0)}{x}+\lim_{x\to0}\frac{f(0)-f(-x)}{x}.$$

In the second expression, we can introduce a new variable $w=-x$:

$$\lim_{x\to0}\frac{f(x)-f(0)}{x}+\lim_{x\to0}\frac{f(0)-f(-x)}{x}=$$

$$=\lim_{x\to0}\frac{f(x)-f(0)}{x}+\lim_{x\to0}\frac{f(-x)-f(0)}{-x}=\lim_{x\to0}\frac{f(x)-f(0)}{x}+\lim_{w\to0}\frac{f(w)-f(0)}{w}.$$

Now recall the definition of the derivative of function f:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Thus,

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}.$$

This is exactly the expression we obtained above. So we can write

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x} + \lim_{w\to 0}\frac{f(w)-f(0)}{w} = f'(0) + f'(0) = 2 * f'(0).$$

Finally, since we are given $f'(0) = 1$, we can say that

$$\lim_{x\to 0}\frac{f(x)-f(-x)}{x} = 2 * 1 = 2.$$

Answer. $\lim_{x\to 0}\frac{f(x)-f(-x)}{x} = 2.$