Sample: Statistics and Probability - Probability Assignment

Problem 1

A $5 gold piece and 14 quarters are in one purse and 15 quarters are in another purse. Five coins are taken from the first purse and placed in the second, and then five coins are taken from the second and placed in the first. What is the probability that after these transactions the gold coin will be found in the second purse?

Solution.

Gold coin will be found in the second purse, in case it will be initially taken in the first (among 5), and then will not be taken from the second.

Probability that the gold coin will be taken form the first equals to

$$p_1 = 1 - \frac{\binom{14}{5}}{\binom{15}{5}} = 1 - \frac{14!}{5!\,9!} \Big/ \frac{15!}{5!\,10!} = 1 - \frac{14!}{5!\,9!} \cdot \frac{5!\,10!}{15!} = 1 - \frac{10}{15} = \frac{1}{3}$$

Probability that the gold coin will not be taken from the second purse:

$$p_2 = \frac{\binom{19}{5}}{\binom{20}{5}} = \frac{19!}{5!\,14!} \cdot \frac{5!\,15!}{20!} = \frac{15}{20} = \frac{3}{4}$$

So the probability that the gold coin will be in the second purse equals to

$$p_1 p_2 = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

Answer. $\frac{1}{4}$

Problem 2

(a) If you hold three tickets to a lottery for which n tickets were sold and 10 prizes are to be given, what is the probability that you will win at least one prize? (b) What is the answer to (a) if you hold only two tickets? (c) Evaluate (a) & (b) for n=100, 000 and n=1,000.

Solution.

(a) There are $\binom{n}{10}$ ways to choose 10 winners among n participants. There are $\binom{n}{10} - \binom{n-3}{10}$ ways to choose winners such that at least one of 3 tickets is winning. So probability of winning equals to

$$\frac{\binom{n}{10}-\binom{n-3}{10}}{\binom{n}{10}} = 1-\frac{\binom{n-3}{10}}{\binom{n}{10}} = 1-\frac{(n-3)!}{10!\,(n-13)!}\cdot\frac{10!\,(n-10)!}{n!}$$
$$= 1-\frac{(n-12)(n-11)(n-10)}{(n-2)(n-1)n} = \frac{30(n^2-12n+44)}{(n-2)(n-1)n}$$

(b) In case of 2 tickets probability of winning equals to

$$1-\frac{\binom{n-2}{10}}{\binom{n}{10}} = 1-\frac{(n-2)!}{10!\,(n-12)!}\cdot\frac{10!\,(n-10)!}{n!} = 1-\frac{(n-11)(n-10)}{(n-1)n} = \frac{10(2n-11)}{(n-1)n}$$

(c) For $n = 1000$:

Probability to win with 3 tickets:

$$\frac{82337}{2769450} \approx 0.0297$$

Probability to win with 2 tickets:

$$\frac{221}{11100} \approx 0.0199$$

For $n = 1000000$:

Probability to win with 3 tickets:

$$\frac{7575666667}{252524494950000} \approx 0.00003$$

Probability to win with 2 tickets:

$$\frac{222221}{11111100000} \approx 0.00002$$

Problem 3

Given $f(x,y) = c(x+y)$ at the point (1,1), (1,2), (2,1), (2,4), and zero elsewhere, (a) evaluate c; (b) find $f(x)$; (c) find the conditional densities $f(y|x)$.

Solution.

(a) Total sum of probabilities equals to 1. Thus

$$f(1,1) + f(1,2) + f(2,1) + f(2,4) = c(2+3+3+6) = 14c = 1$$

$$c = \frac{1}{14}$$

(b) $f(x)$ is non-zero for $x = 1$ and $x = 2$.

$$f(1) = f(1,1) + f(1,2) = \frac{1}{14}(2+3) = \frac{5}{14}$$

$$f(2) = f(2,1) + f(2,4) = \frac{1}{14}(3+6) = \frac{9}{14}$$

(c) $f(y|x)$ are calculated using formula

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$f(y=1|x=1) = \frac{14}{5} \cdot \left(\frac{1}{14}(1+1)\right) = \frac{2}{5}$$

$$f(y=2|x=1) = \frac{14}{5} \cdot \left(\frac{1}{14}(1+2)\right) = \frac{3}{5}$$

$$f(y=1|x=2) = \frac{14}{9}\left(\frac{1}{14}(2+1)\right) = \frac{1}{3}$$

$$f(y=4|x=2) = \frac{14}{9}\left(\frac{1}{14}(2+4)\right) = \frac{2}{3}$$

Problem 4

If X has a (marginal) Poisson distribution with mean μ and, given $X = x$, Y has a (conditional) binomial distribution with mean xp, show that the (marginal) distribution of Y is Poisson with mean μp.

Solution.

Let's find marginal distribution of Y

$$P(Y = y) = \sum_{x=0}^{\infty} P(Y = y|X = x)P(X = x)$$

Since $P(Y = y|X = x)$ equals to 0 for $y < x$ (because random variable with binomial distribution with parameters (x, p) reaches values from 0 to x),

$$P(Y = y) = \sum_{x=y}^{\infty} P(Y = y|X = x)P(X = x)$$

Since Y has binomial distribution with parameters (n, p) we have:

$$P(Y = y|X = x) = \binom{x}{y} p^y (1-p)^{x-y}$$

Since X has Poisson distribution with mean μ,

$$P(X = x) = \frac{\mu^x}{x!} e^{-\mu}$$

Thus

$$P(Y = y) = \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \cdot \frac{\mu^x}{x!} e^{-\mu} = p^y e^{-\mu} \sum_{x=y}^{\infty} \frac{x!}{y!\,(x-y)!} (1-p)^{x-y} \cdot \frac{\mu^x}{x!}$$
$$= \frac{p^y e^{-\mu}}{y!} \sum_{x=y}^{\infty} \frac{\mu^x (1-p)^{x-y}}{(x-y)!}$$

Let's change summation variable by $x \to x - y$

$$P(Y = y) = \frac{p^y e^{-\mu}}{y!} \sum_{x=0}^{\infty} \frac{\mu^{x+y}(1-p)^x}{x!} = \frac{p^y e^{-\mu}}{y!} \mu^y \sum_{x=0}^{\infty} \frac{\mu^x (1-p)^x}{x!}$$
$$= \frac{p^y e^{-\mu}}{y!} \mu^y \sum_{x=0}^{\infty} \frac{\left(\mu(1-p)\right)^x}{x!}$$

The sum in the last expression is the Taylor expansion of e^z for $z = \mu(1-p)$. Thus

$$P(Y = y) = \frac{p^y e^{-\mu}}{y!} \mu^y e^{\mu(1-p)} = \frac{(\mu p)^y e^{-\mu p}}{y!}$$

So Y has Poisson distribution with mean μp.

Problem 5

Given the continuous density $f(x) = cxe^{-x}, x > 0$, (a) determine the value of c; (b) find $P(X < 3)$; (c) find $P(3 < X < 4)$.

Solution.

(a) Let's determine constant c from the equality

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} cxe^{-x}dx = (integration\ by\ parts) = c\left(-xe^{-x}|_0^\infty - \int_0^\infty -e^{-x}dx\right)$$
$$= c(-e^{-x}|_0^\infty) = c = 1$$

So $c = 1$.

(b) $P(X < 3) = \int_{-\infty}^{3} f(x)dx = \int_0^3 xe^{-x}dx = -xe^{-x}|_0^3 - \int_0^3 -e^{-x}dx = -3e^{-3} - e^{-x}|_0^3 = 1 - 4e^{-3}$

(c) $P(3 < X < 4) = \int_3^4 f(x)dx = \int_3^4 xe^{-x}dx = -xe^{-x}|_3^4 - e^{-x}|_3^4 = -4e^{-4} + 3e^{-3} - e^{-4} + e^{-3} = 4e^{-3} - 5e^{-4}$