Sample: Differential Geometry - Mathematics Assignment

Question 1 . Steiner's Roman surface is defined as the image of the map

$$F: \mathbf{RP}_2 \to \mathbf{R}^3$$

induced by the map $\hat{F}: \mathbf{S}^2 \to \mathbf{R}^3$ such that

$$\hat{F}(x_1, x_2, x_3) = (x_2 x_3, x_1 x_3, x_1 x_2).$$

Show that F fails to be an immersion at six points on $\mathbf{RP}_2$.

Proof. Let

$$\mathbf{S}^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$$

be the unit sphere in $\mathbf{R}^3$. Then by definition the projective plane $\mathbf{RP}_2$ is the space of pairs of antipodal points of $\mathbf{S}^2$, that is the factor-space of $\mathbf{S}^2$ by the following equivalence relation:

$$(x_1, x_2, x_3): (-x_1, -x_2, -x_3).$$

Let us prove that $\hat{F}$ induces a certain map $\mathbf{RP}_2 \to \mathbf{R}^3$.

Let $\alpha: \mathbf{S}^2 \to \mathbf{RP}_2$ be the factor map.

Since $(-x_i)(-x_j) = x_i x_j$ it follows that

$$\begin{aligned} \hat{F}(-x_1, -x_2, -x_3) &= (-x_2(-x_3), -x_1(-x_3), -x_1(-x_2)) \\ &= (x_2 x_3, x_1 x_3, x_1 x_2) \\ &= \hat{F}(x_1, x_2, x_3). \end{aligned}$$

Thus $\hat{F}$ constant of equivalence class, and so it induces a map $F: \mathbf{RP}_2 \to \mathbf{R}^3$ such that $\hat{F} = F \circ \alpha$.

Now let us check that F is an immersion. First we recall the definition of an immersion.

Let M, N are smooth two manifolds, $f: M \to N$ be a C^1 map, and $x \in M$. Then f is an *immersion* at x if the tangent map $T_x f: T_x M \to T_{f(x)} N$ is injective. Suppose $\dim M = m$ and $\dim N = n$, and we choose local coordinates $(x_1, \dots, x_m)$ on M at x and $(y_1, \dots, y_n)$ on N at $f(x)$. Then f is an immersion at x if the Jacobi matrix of f at x (consisting of partial derivatives of coordinate functions of f) has rank m.

Evidently, a composition of immersions is an immersion as well.

Notice that the factor map $\alpha: \mathbf{S}^2 \to \mathbf{RP}_2$ is local diffeomorphism, so the tangent map α at each point $q \in \mathbf{S}^2$ is an isomorphism, and so α is an immersion. Thus in order to find points on $\mathbf{RP}_2$ at which F is not an immersion we should find points on $\mathbf{S}^2$ at which $\hat{F}$ is not an immersion and take their images in $\mathbf{RP}_2$.

Moreover, we can extend $\hat{F}$ to the map $\mathbf{R}^3 \to \mathbf{R}^3$ by the same formula. Then the Jacobi matrix of $\hat{F}$ is equal to

$$J(\hat{F}) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$$

and its determinant is

$$|J(\hat{F})| = \begin{vmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{vmatrix} = -x_1 x_2 x_3 - x_1 x_2 x_3 = -2 x_1 x_2 x_3.$$

Let $q = (x_1, x_2, x_3)$. Then $|J(\hat{F})(q)| \neq 0$ if and only if all coordinates (x_1, x_2, x_3) are non-zero, i.e. the point q does not belongs to the coordinate planes xy, yz, and xz. At each of these points the tangent map

$$T_q \hat{F}: T_q \mathbf{R}^3 \to T_{\hat{F}(q)} \mathbf{R}^3$$

is an isomorphism. In particular, if in addition $q \in \mathbf{S}^2$, the restriction of $T_q \hat{F}$ to the tangent plane $T_q \mathbf{S}^2$ is injective, whence $\hat{F}$ is an immersion at q. Therefore at the corresponding point $\alpha(q) \in \mathbf{RP}_2$ the map F is an immersion as well.

Suppose one of coordinates of q is zero. Not loosing generality assume that $x_1 = 0$. As $x_1^2 + x_2 + x_3^2 = 1$, it follows that $x_2^2 + x_3^2 = 1$, whence either x_2 or x_3 is non-zero.
Then the Jacobi matrix at q is

$$J(\hat{F})(q) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}$$

and its rank (as of a map $\mathbf{R}^3 \to \mathbf{R}^3$) is 2, as at least one of the following 2×2-minores is non-zero:

$$\begin{vmatrix} 0 & x_3 \\ x_3 & 0 \end{vmatrix} = -x_3^2, \qquad \begin{vmatrix} 0 & x_2 \\ x_2 & 0 \end{vmatrix} = -x_2^2.$$

Now the us find intersection of the null space of matrix $J(\hat{F})(q)$ with the tangent space $T_q\mathbf{S}^2$. Then the restriction of $\hat{F}$ to $\mathbf{S}^2$ is an immersion if and only if that intersection is non-zero.
Suppose the tangent vector $\xi = (a, b, c) \in T_q\mathbf{R}^3$ belongs to the null space of $J(\hat{F})(q)$. Thus

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = J(\hat{F})(p) \cdot \xi = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} bx_3 + cx_2 \\ ax_2 \\ ax_3 \end{pmatrix}.$$

As either x_2 or x_3 is non-zero, it follows that $a = 0$ and $bx_3 + cx_2 = 0$. Whence the null space of $J(\hat{F})(q)$ is spanned by the following vector

$$\eta = \begin{pmatrix} 0 \\ -x_2 \\ x_3 \end{pmatrix}$$

The restriction of $\hat{F}$ to $\mathbf{S}^2$ at q is not an immersion if and only if η belongs to the tangent space $T_q\mathbf{S}^2$ of $\mathbf{S}^2$ at q. The latter condition means that η is orthogonal to the vector $\vec{q} \in \mathbf{R}^3$, so their scalar product is zero:

$$\langle \eta, \vec{q} \rangle = 0 = (0, -x_2, x_3) \cdot (0, x_2, x_3) = 0 \cdot 0 - x_2x_2 + x_3x_3 = -x_2^2 + x_3^2.$$

It then follows that

$$x_2^2 = x_3^2, \qquad \Rightarrow \qquad x_2 = \pm x_3.$$

As $x_2^2 + x_3^2 = 1$, we obtain that

$$x_2^2 = x_3^2 = \frac{1}{2}, \qquad \Rightarrow \qquad x_2 = \pm\frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{\sqrt{2}}.$$

Thus there are the 4 points on $\mathbf{S}^2$ with x_0 at which $\hat{F}$ is not an immersion:

$$X_1 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad X_2 = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$X_3 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \qquad X_4 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Since $X_1 = -X_2$ they define the same point $\alpha(X_1) = \alpha(X_2)$ on $\mathbf{RP}_2$, and at this point the map F is not an immersion. The same statement hold for the pair X_3 and X_4.
Thus we have found two points on PR^2 with $x_1 = 0$ at which F is not an immersion.
Due to the symmetry, in each of the cases $x_2 = 0$ and $x_3 = 0$ we also have 2 non-immersion points, and so the map F has the following six points at which it is not an immersion:

$$\pm\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \pm\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\pm\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \pm\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad \pm\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$

Question 2. Consider the map $\eta: \mathbf{S}^2 \to \mathbf{R}^4$ such that

$$\eta(u,v,w) = (u^2 - v^2, uv, uw, vw),$$

where all points (u,v,w) on the sphere satisfy $u^2+v^2+w^2 = 1$. Show that $\eta(u,v,w) = \eta(u',v',w')$ if and only if $(u,v,w) = \pm(u',v',w')$, hence η defines a one-to one map from $\mathbf{RP}_2$ to its image in $\mathbf{R}^4$. Show also that the image of η is a *proper* subset of $F^{-1}(0)$ for the map $F: \mathbf{R}^4 \to \mathbf{R}^2$ such that

$$F(x,y,z,t) = (y(z^2-t^2) - xzt, y^2z^2 + y^2t^2 + z^2t^2 - yzt).$$

Proof. Let $(u,v,w),(u',v',w') \in \mathbf{S}^2$. If $(u,v,w) = (u',v',w')$, then trivially $\eta(u,v,w) = \eta(u',v',w')$. Also if $(u,v,w) = -(u',v',w')$, then

$$\begin{aligned}\eta(u',v',w') &= \eta(-u,-v,-w)\\ &= ((-u)^2-(-v)^2,(-u)(-v),(-u)(-w),(-v)(-w))\\ &= (u^2-v^2,uv,uw,vw)\\ &= \eta(u,v,w).\end{aligned}$$

Conversely, suppose $\eta(u,v,w) = \eta(u',v',w')$. Then we have the following equalities:

$$u^2 - v^2 = u'^2 - v'^2,$$

$$uv = u'v',$$

$$uw = u'w',$$

$$vw = v'w'.$$

Notice that

$$(u^2-v^2)^2 + 4(uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4 = (u^2+v^2)^2,$$

whence from $u^2 - v^2 = u'^2 - v'^2$ and $uv = u'v'$ we obtain

$$u^2 + v^2 = u'^2 + v'^2.$$

Adding this to $u^2 - v^2 = u'^2 - v'^2$ we get

$$2u^2 = 2u'^2, \qquad \Rightarrow \qquad u = \pm u'.$$

Therefore

$$v^2 = v'^2, \qquad \Rightarrow \qquad v = \pm v'.$$

Since $(u,v,w),(u',v',w') \in \mathbf{S}^2$, we have that

$$u^2+v^2+w^2 = 1 = u'^2+v'^2+w'^2 = 1,$$

and so

$$w = \pm w'.$$

Thus

$$u = \alpha u', \qquad v = \beta v', \qquad w = \gamma w'$$

for some $\alpha,\beta,\gamma = \pm 1$.

We claim that one can always assume that $\alpha = \beta = \gamma$. Consider two cases.

1) Suppose there are two non-zero coordinates, say $u, v \neq 0$. Then the corresponding coefficients coincides $\alpha = \beta$. Indeed,

$$uv = u'v' = \alpha u \beta v, \qquad \Rightarrow \qquad 1 = \alpha\beta, \qquad \Rightarrow \qquad \alpha = \beta.$$

Now if $w = 0$, then $w' = \gamma w = 0$, and so

$$(u',v',w') = (\alpha u, \alpha v, 0) = \alpha \cdot (u,v,0) = \alpha \cdot (u,v,w).$$

If $w \neq 0$, then $\alpha = \beta = \gamma$.

2) Suppose two of coordinates (u,v,w) are zero, say, let $v = w = 0$, and $u \neq 0$. Then $v' = w' = 0$, and $u' = \pm u$, so

$$(u', v', w') = (\pm u, 0,0) = \pm(u, 0,0) = \pm(u, v, w).$$

Thus $\eta(u, v, w) = \eta(u', v', w')$ if and only if $(u, v, w) = \pm(u', v', w')$.

Let us prove that the image of η is a proper subset of $F^{-1}(0)$ for the map $F: \mathbf{R}^4 \to \mathbf{R}^2$ defined by

$$F(x, y, z, t) = (y(z^2 - t^2) - xzt, y^2z^2 + y^2t^2 + z^2t^2 - yzt).$$

It suffices to prove that $F \circ \eta: \mathbf{S}^2 \to \mathbf{R}^2$ a constant map equal to 0. Indeed, since $u^2 + v^2 + w^2 = 1$, we obtain that

$$\begin{aligned}
F \circ \eta(u, v, w) &= F(u^2 - v^2, uv, uw, vw) \\
&= (uv((uw)^2 - (vw)^2) - (u^2 - v^2)uwvw, \\
&\qquad (uv)^2(uw)^2 + (uv)^2(vw)^2 + (uw)^2(vw)^2 - uvuwvw) \\
&= (u^3vw^2 - uv^3w^2 - u^3vw^2 + uv^3w^2, \\
&\qquad u^4v^2w^2 + u^2v^4w^2 + u^2v^2w^4 - u^2v^2w^2) \\
&= (0, (v^2 + u^2 + w^2 - 1)u^2v^2w^2) = (0,0).
\end{aligned}$$