Sample: Calculus - Investigation of Functions

Sketch the functions:

i. $f(x) = \frac{x}{x^2-9}$
ii. $f(x) = 2x^3 - 3x^2 - 12x$.

Solution:

i. $f(x) = \frac{x}{x^2-9}$.

1. Find the domain of function. Domain will be $(-\infty, \infty)$ except points where
$$x^2 - 9 = 0$$
$$x_{1,2} = \pm 3$$
So domain of function is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.
2. Find the $y-$ intercept by substituting $x = 0$:
$$f(x) = \frac{0}{0-9} = 0.$$
So, the $y-$ intercept is $(0, 0)$.
Find the $x-$ intercept by substituting $y = 0$:
$$0 = \frac{x}{x^2 - 9}$$
$$x = 0$$
So, the $x-$ intercept is $(0, 0)$.
3. Find any vertical asymptotes by investigating where the denominator is 0:
$$x^2 - 9 = 0$$
$$x_{1,2} = \pm 3$$
So, the vertical asymptotes are: $x = -3$ and $x = 3$.
Find any horizontal asymptotes by finding the limits as $x \to -\infty$ and $x \to \infty$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x^2 - 9} = 0$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2 - 9} = 0$$
So, the horizontal asymptote is $y = 0$
4. Investigate symmetry
$$f(-x) = \frac{-x}{(-x)^2 - 9} = -\frac{x}{x^2 - 9} = -f(x)$$
So, function is odd and graph is symmetric about the origin.
5. Find $f'(x)$.
$$f'(x) = \frac{x^2 - 9 - x \cdot 2x}{(x^2 - 9)^2} = -\frac{x^2 + 9}{(x^2 - 9)^2}$$
Locate any critical points by solving the equation $f'(x) = 0$
$$-\frac{x^2 + 9}{(x^2 - 9)^2} = 0$$
There aren't any x when $f'(x) = 0$, so there aren't any critical points.
Determine where $f(x)$ is increasing or decreasing:
As $f'(x) < 0$ for any x from domain, then function is decreasing on any x from domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
6. Find $f''(x)$
$$f''(x) = \frac{-2x(x^2 - 9)^2 + (x^2 + 9) \cdot 4x(x^2 - 9)}{(x^2 - 9)^4} = \frac{2x(x^2 + 27)}{(x^2 - 9)^3}$$

Locate potential inflection points by solving the equation $f''(x) = 0$

$$\frac{2x(x^2+27)}{(x^2-9)^3} = 0$$

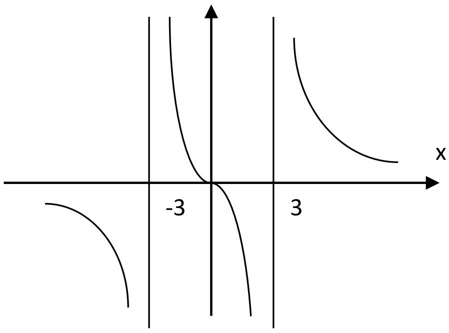
$$x = 0$$

Determine where $f(x)$ is concave upward or concave downward:

$f''(x) > 0$, when $x \in (-3,0) \cup (3,\infty)$, so on this intervals function is concave upward

$f''(x) < 0$, when $x \in (-\infty,-3) \cup (0,3)$, so on this intervals function is concave downward

7. Sketch the graph



ii. $f(x) = 2x^3 - 3x^2 - 12x$

1. Find the domain of function. Domain will be $(-\infty,\infty)$.
2. Find the $y-$intercept by substituting $x = 0$:

   $f(x) = 2\cdot 0 - 3\cdot 0 - 12\cdot 0 = 0.$

   So, the $y-$intercept is $(0,0)$.

   Find the $x-$intercept by substituting $y = 0$:

$$0 = 2x^3 - 3x^2 - 12x$$

$$x(2x^2 - 3x - 12) = 0$$

$$x_1 = 0$$

$$x_2 = \frac{1}{4}\left(3 - \sqrt{105}\right) \approx -1.81$$

$$x_3 = \frac{1}{4}\left(3 + \sqrt{105}\right) \approx 3.31$$

   So, the $x-$intercept are $(0,0), (-1.81,0), (3.31,0)$.

3. Find any vertical asymptotes by investigating where the denominator is 0:

   So, there aren't any vertical asymptotes.

   Find any horizontal asymptotes by finding the limits as $x \to -\infty$ and $x \to \infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^3 - 3x^2 - 12x = -\infty$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x^3 - 3x^2 - 12x = \infty$$

   So, there aren't any horizontal asymptotes

4. Investigate symmetry

$$f(-x) = 2(-x)^3 - 3(-x)^2 - 12(-x) = -2x^3 - 3x^2 + 12x \neq -f(x) \; or \; f(x)$$

   So, function is neither odd or even, and graph isn't symmetric about the origin or $y-$axis.

5. Find $f'(x)$.

$$f'(x) = 6x^2 - 6x - 12$$

Locate any critical points by solving the equation $f'(x) = 0$

$$6x^2 - 6x - 12 = 0$$
$$x_1 = -1$$
$$x_2 = 2$$

So, there are 2 critical points $x = -1$ and $x = 2$.

$$f(-1) = 2 \cdot (-1)^3 - 3 \cdot (-1)^2 - 12 \cdot (-1) = -2 - 3 + 12 = 7$$
$$f(2) = 2 \cdot 2^3 - 3 \cdot 2^2 - 12 \cdot 2 = -20$$

Determine where $f(x)$ is increasing or decreasing:

$f'(x) > 0$, when $x \in (-\infty, -1) \cup (2, \infty)$, so on this intervals function is increasing

$f'(x) < 0$, when $x \in (-1, 2)$, so on this intervals function is concave decreasing

From intervals of increasing/decreasing show then $x = -1$ is a local maximum and $x = 2$ is a local minimum.

6. Find $f''(x)$

$$f''(x) = 12x - 6$$

Locate potential inflection points by solving the equation $f''(x) = 0$

$$12x - 6 = 0$$
$$x = 0.5$$

So, $x = 0.5$ is the inflection point

$$f(0.5) = 2 \cdot (0.5)^3 - 3 \cdot (0.5)^2 - 12 \cdot (0.5) = -6.5$$

Determine where $f(x)$ is concave upward or concave downward:

$f''(x) > 0$, when $x \in (0.5, \infty)$, so on this intervals function is concave upward

$f''(x) < 0$, when $x \in (-\infty, 0.5)$, so on this intervals function is concave downward

7. Sketch the graph