Sample: Calculus - Inequalities

Question 1

Suppose that a function $f(x)$ defined everywhere is decreasing and concave down. Prove that there is a number a, such that $f(a) < 0$.

Solution.

Since function is decreasing, for $x_0 < y_0$ we have: $f(y_0) < f(x_0)$. Since function is concave down, for $z_0 > y_0$ we have:

$$f(y_0) \geq \frac{z_0 - y_0}{z_0 - x_0} f(x_0) + \frac{y_0 - x_0}{z_0 - x_0} f(z_0)$$

So

$$\begin{aligned} f(z_0) &\leq \frac{1}{y_0 - x_0}\big(f(y_0)(z_0 - x_0) - (z_0 - y_0) f(x_0)\big) \\ &= \frac{1}{y_0 - x_0}\Big(z_0\big(f(y_0) - f(x_0)\big) + \big(y_0 f(x_0) - f(y_0) x_0\big)\Big) \\ &= z_0 \frac{f(y_0) - f(x_0)}{y_0 - x_0} + \frac{y_0 f(x_0) - f(y_0) x_0}{y_0 - x_0} \end{aligned}$$

The last expression is a linear function of z_0. Slope of this function is $\frac{f(y_0) - f(x_0)}{y_0 - x_0} < 0$ because $y_0 > x_0$ and $f(y_0) < f(x_0)$. This function intercepts x-axis at some point p_0.

Thus $\forall x > p_0$ this function is negative. Since $f(z_0)$ is less than values of this function, there exists a: $f(a) < 0$.

Question 2

Find without a calculator or computer the integral part of

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{1{,}000{,}000}}$$

(The integral part means the largest integer less or equal to the given one).

Solution.

Consider a function

$$f(x)=\frac{1}{\sqrt{x}}$$

$$\int_1^{1000000} f(x)dx = 2x^{\frac{1}{2}}\Big|_{x=1}^{1000000} = 2000-2=1998$$

Since function $\frac{1}{\sqrt{x}}$ is monotone decreasing, we have:

$$\sum_{n=2}^{1000000}\frac{1}{\sqrt{n}}=\sum_{n=1}^{1000000}\frac{1}{\sqrt{n}}-1<\int_1^{1000000} f(x)dx<\sum_{n=1}^{999999}\frac{1}{\sqrt{n}}$$
$$=\sum_{n=1}^{1000000}\frac{1}{\sqrt{n}}-\frac{1}{1000}$$

So

$$\frac{1}{1000}+\int_1^{1000000} f(x)dx<\sum_{n=1}^{1000000}\frac{1}{\sqrt{n}}<\int_1^{1000000} f(x)dx+1$$

Thus

$$1998.001<\sum_{n=1}^{1000000}\frac{1}{\sqrt{n}}<1999$$

So integral part of

$$\sum_{n=1}^{1000000}\frac{1}{\sqrt{n}}$$

equals 1998.

Question 3

Prove without calculator or computer that $2015^{2013} < 2014^{2014} < 2013^{2015}$.

Solution.

Consider a function

$$f(x) = (2014 + x)^{2014-x}$$

We need to prove

$$f(1) < f(0) < f(-1)$$

Let's find derivative $f'(x)$:

$$f'(x) = \left(e^{\ln(2014+x)\cdot(2014-x)}\right)' = (2014 + x)^{2014-x}\left(\frac{2014 - x}{2014 + x} - \ln(2014 + x)\right)$$

For $x \in [-2,2]$ we have:

$$(2014 + x)^{2014-x} \geq 0$$

$$\ln 2012 \leq \ln(2014 + x) \leq \ln 2014$$

$$\frac{2012}{2016} \leq \frac{2014 - x}{2014 + x} \leq \frac{2016}{2012}$$

So for $x \in [-2,2]$ we have: $f'(x) < 0$.

Thus f is decreasing on $[-2,2]$. So

$$f(-1) > f(0) > f(1)$$

The inequality is proved.

Question 4

Prove that for any $n > 1$

$$\frac{1}{2\sqrt{n}} < \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\dots\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}.$$

Solution.

We need to prove:

$$\frac{1}{2\sqrt{n}} < \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\dots\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$$

Let's take logarithm:

Consider a function

$$f(x) = \ln\left(\frac{2x-1}{2x}\right)$$

This function is monotone increasing.

Thus

$$\sum_{k=1}^{n-1} \ln\left(\frac{2k-1}{2k}\right) < \int_1^n f(x)dx < \sum_{k=2}^{n} \ln\left(\frac{2k-1}{2k}\right)$$

$$-\ln\left(\frac{2n-1}{2n}\right) + \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right) < \int_1^n f(x)dx < \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right) - \ln\frac{1}{2}$$

So

$$\int_1^n f(x)dx + \ln\frac{1}{2} < \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right) < \int_1^n f(x)dx + \ln\left(\frac{2n-1}{2n}\right)$$

Note that

$$\sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right) = \ln\prod_{k=1}^{n}\frac{2k-1}{2k}$$

$$\int_1^n f(x)dx = x\ln\left(\frac{2x-1}{2x}\right) + \frac{1}{2}\ln(2x-1)\Big|_{x=1}^{n}$$
$$= \left(n\ln\frac{2n-1}{2n} - \frac{1}{2}\ln(2n-1)\right) - \left(\ln\frac{1}{2}\right)$$

So

$$n\ln\frac{2n-1}{2n} - \frac{1}{2}\ln(2n-1) < \sum_{k=1}^{n}\ln\left(\frac{2k-1}{2k}\right)$$
$$< \left(n\ln\frac{2n-1}{2n} - \frac{1}{2}\ln(2n-1)\right) - \left(\ln\frac{1}{2}\right) + \ln\left(\frac{2n-1}{2n}\right)$$

Taking exponent from both sides we get:

$$\frac{1}{\sqrt{2n-1}}\left(\frac{2n-1}{2n}\right)^n < \prod_{k=1}^{n}\frac{2k-1}{2k} < \frac{1}{\sqrt{2n-1}}\left(\frac{2n-1}{2n}\right)^n \cdot \frac{2n-1}{n}$$

Now,

$$\frac{1}{\sqrt{2n-1}}\left(\frac{2n-1}{2n}\right)^n > \frac{1}{2\sqrt{n}}$$

and

$$\frac{1}{\sqrt{2n-1}}\left(\frac{2n-1}{2n}\right)^n \cdot \frac{2n-1}{n} < \frac{1}{\sqrt{2n}}$$

So the inequality holds.