Sample: Differential Calculus Equations - Fourier Transform

Question 1. Find $u(x)$ if $\int_0^\infty u(x-\xi)e^{-\xi}d\xi = f(x)$.

Let's take Fourier transform of both parts of the equation.

$$F\left(\int_0^\infty u(x-\xi)e^{-\xi}d\xi\right) = F(f)$$

$\int_0^\infty u(x-\xi)e^{-\xi}d\xi$ is convolution of 2 functions: $u(x)$ and e^{-x}. Fourier transform of convolution equals to product of Fourier transforms of functions.

$$F(u)F(e^{-x}) = F(f)$$

$$F(u) = F(f)(1+iw)$$

$$u = F^{-1}\big(F(f) + iwF(f)\big) = F^{-1}\big(F(f)\big) + F^{-1}\big(iwF(f)\big) = f + f'$$

So,

$$u(x) = f(x) + f'(x)$$

Question 2. Find the solution explicitly in the case $f(x) = x$ and verify that it works.

Let's check the solution for $f(x) = x$:

$$u(x) = f(x) + f'(x) = x + 1$$

$$\int_0^\infty u(x-\xi)e^{-\xi}d\xi = \int_0^\infty (x-\xi+1)e^{-\xi}d\xi = (x+1)\int_0^\infty e^{-\xi}d\xi - \int_0^\infty \xi e^{-\xi}d\xi$$
$$= (x+1)\left(-e^{-\xi}\right)\Big|_{\xi=0}^{\xi=\infty} - \left(-\xi e^{-\xi} - e^{-\xi}\right)\Big|_{\xi=0}^{\xi=\infty} = (x+1) - 1 = x = f(x)$$

So the equality holds.

Question 3. Find the solution in the case $f(x) = e^{kx},\ \ k > -1$ and verify that it works.

$$f(x) = e^{kx}$$

$$u(x) = f(x) + f'(x) = e^{kx} + ke^{kx} = (k+1)e^{kx}$$

$$\int_0^\infty (k+1)e^{k(x-\xi)}e^{-\xi}d\xi = (k+1)e^{kx}\int_0^\infty e^{-(k+1)\xi}d\xi = (k+1)e^{kx}\left(\frac{e^{-(k+1)\xi}}{-(k+1)}\right)\Bigg|_{\xi=0}^{\xi=\infty} = e^{kx} = f(x)$$

The equality holds.

Question 4. Solve the equation in (1) without using Fourier transforms.

$$\int_0^{\infty} u(x-\xi)e^{-\xi}d\xi = f(x)$$

Let's make change of variables in the equation (y is a new variable of integration):

$$\int_0^{\infty} u(x-\xi)e^{-\xi}d\xi = \begin{vmatrix} \xi = x - y \\ d\xi = -dy \end{vmatrix} = \int_x^{-\infty} u(y)e^{y-x}(-1)\,dy = \int_{-\infty}^{x} u(y)e^{y-x}\,dy = f(x)$$

Term e^{-x} can be moved out the integral (it does not depend on y):

$$e^{-x}\int_{-\infty}^{x} u(y)e^{y}dy = f(x)$$

$$\int_{-\infty}^{x} u(y)e^{y}dy = e^{x}f(x)$$

Let's differentiate both parts of the equation by x:

$$u(x)e^{x} = \left(e^{x}f(x)\right)' = e^{x}f(x) + e^{x}f'(x)$$

$$u(x) = f(x) + f'(x)$$