Sample: Electromagnetism - Electromagnetism and Materials

Question 1

A disk of radius R and negligible thickness exists in free space, with no other charges nearby.

(a) The upper surface of the disk is charged with a uniform surface charge distribution given by $\rho_s = a$. Find the total charge on the disk surface in terms of a and R.
**Solution:** since the charge is uniformly distributed on the surface of the disk, we can simply write that $Q = \rho_s \cdot S$, where S is the area of the disk. Area of the disk is $S = \pi \cdot R^2$, so the total charge on the surface is $Q = \pi a R^2$.
Answer: $Q = \pi a R^2$.

(b) The charge distribution described above now changes to $\rho_s = a\left(1 - \frac{x}{R}\right)$, where x is the radial distance from the center of the disk.

(i) Sketch, on the same axes, both the old and the new charge densities as a function of distance from the center of the disk.
Answer:
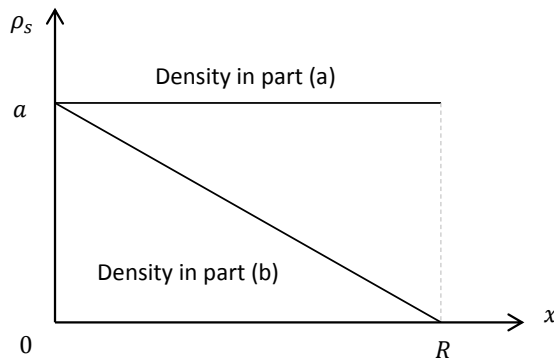
(ii) Write down the integral required to find the total charge on the disk surface with this new distribution.
Solution: let us denote the element of the surface of the disk dS and region of integration D. So the integral takes form $I = \iint \rho_s dS$. In the polar system we can rewrite it as $I = \iint \rho_s x\,dx\,d\phi$, here x means the radial distance from the center. We are integrating over the surface of the disk of radius R, so the upper limit of integration is R and the lower is 0:

$$I = \int_0^R x\,dx \int_0^{2\pi} \rho_s\,d\phi.$$

Answer: $I = \int_0^R x\,dx \int_0^{2\pi} \rho_s\,d\phi$.

(iii) Solve the integral to find the total charge.
Solution: $Q = \int_0^R x\,dx \int_0^{2\pi} \rho_s\,d\phi = \int_0^R x\,dx \int_0^{2\pi} a\left(1 - \frac{x}{R}\right)d\phi = 2\pi a \int_0^R \left(x - \frac{x^2}{R}\right)dx$

Now we are integrating over the radial variable x:

$$Q = 2\pi a \left(\frac{x^2}{2} - \frac{x^3}{3R}\right)\Bigg|_0^R = 2\pi a \left(\frac{R^2}{2} - \frac{R^3}{3R}\right) = 2\pi a \frac{R^2}{6} = \frac{\pi a R^2}{3}.$$

Answer: $Q = \frac{\pi a R^2}{3}$.

(iv) Compare your answer to that obtained in part (a).

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Solution: in the first case $Q = Q_1 = \pi a R^2$, in the second $Q = Q_2 = \frac{\pi a R^2}{3}$. So we can conclude that $Q_1 > Q_2$. Q_2 is smaller because the charge in part (b) is non-uniformly distributed over the surface of the disk and the charge density in each point at the distance x is always smaller than the charge density in part (a).

Answer: Total charge in part (b) is smaller than total charge in part (a).

Question 2

(a) An insulating sphere of radius R is uniformly charged with charge Q. The permittivity is ϵ_0 everywhere.

(i) Draw a simple sketch of this situation, showing the electric field lines.

Answer: because of spherical symmetry of this system, electric field is directed along the radius from the center of the sphere with a magnitude defined by the magnitude of the radius.
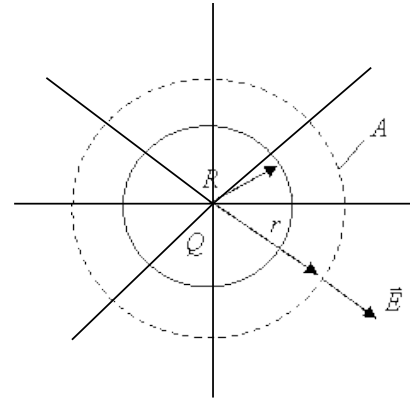
(ii) Find the volume charge density ρ_V.

Solution: since the charge is distributed uniformly and the volume of the sphere is $V = \frac{4}{3}\pi R^3$.

$$\rho_V = \frac{Q}{V} = \frac{3Q}{4\pi R^3}.$$

Answer: $\rho_V = \frac{3Q}{4\pi R^3}$.

(iii) Use Gauss' law to find the electric field both inside and outside the sphere. State explicitly which coordinate system you have used. Indicate your Gaussian surfaces on your sketch.

Solution: we will use the spherical coordinate system to determine the electric field. The Gaussian surface is in this case a sphere with radius r (indicated A on the sketch). We will consider two cases: $r < R$ and $r > R$. In the first case

$$\int E dS = E(r)4\pi r^2 = \frac{q}{\epsilon_0}$$

Charge q restricted by the Gaussian surface is $q = \rho_V \cdot V = \frac{3Q}{4\pi R^3} \cdot \frac{4\pi r^3}{3} = Q\frac{r^3}{R^3}$. Now we are substituting it in the upper equation:

$$E(r)4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3} \quad \rightarrow \quad E(r) = \frac{Q \cdot r}{4\pi\epsilon_0 R^3}, \quad r \leq R.$$
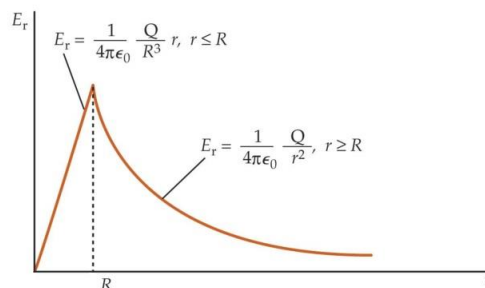
In the second case restricted by the Gaussian surface charge $q = Q$. So we become:

$$\int E dS = E(r)4\pi r^2 = \frac{Q}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r \geq R.$$

Answer: $E(r) = \frac{Q \cdot r}{4\pi\epsilon_0 R^3}, r \leq R$ and $E(r) = \frac{Q}{4\pi\epsilon_0 r^2}, r \geq R$.

(iv) Sketch a graph of the variation in the magnitude of the electric field with radial distance.
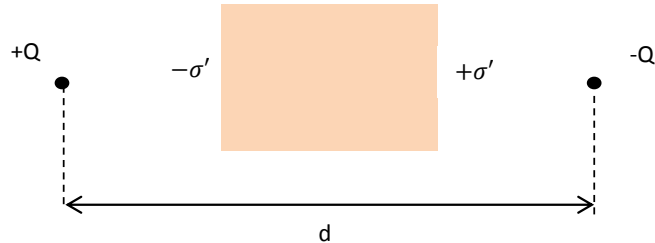
Answer:

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(b) Two point charges of equal and opposite charge are separated by a distance d in free space. A dielectric slab of constant permittivity is inserted between the charges without touching either of them.

    (i)   Draw a diagram of this situation. Briefly describe the bound surface and volume charges in the slab.

        Solution:

In this situation there are bound surface charges (with some difficult surface charge distribution σ'). The left side of the slab that is closer to the positive charge has the negative bound charges and the right side that is closer to the negative charge has the positive ones. But in this case there are no bound volume charges (because of constant permittivity). Volume density of bound charges we can find as a divergence of the polarization vector

$$\rho' = -\nabla \cdot \vec{P}$$

In this case $\vec{P} = \epsilon\epsilon_0\vec{E}$, so $\rho' = -\epsilon\epsilon_0\nabla \cdot \vec{E}$. But inside the slab there are no free charges, so $\nabla \cdot \vec{E} = 0$ and hence $\rho' = 0$.

Answer: bound surface charges would be nonzero, but there would be no bound volume charges.

    (ii)   Does the force between the charges change when the slab is inserted? Justify your answer.

        Solution: we can find the force between the charges when the slab is inserted from some simple considerations. Without the slab the force of attraction between two charges is

$$F = \frac{Q^2}{4\pi\epsilon_0 d^2}$$

If the charges are inserted in the medium with dielectric constant ϵ (remember that $\epsilon > 1$), the force of attraction is

$$F = \frac{Q^2}{4\pi\epsilon_0 \epsilon d^2}$$

Last equation means that two charges must be kept in air at the distance $d' = \sqrt{\epsilon}d$ to keep the force of interaction the same. When the slab of thickness b is inserted the effective distance between the charges should be $d_{eff} = d - b + \sqrt{\epsilon}b = d + (\sqrt{\epsilon} - 1)b$. So the force of interaction is

$$F = \frac{Q^2}{4\pi\epsilon_0\big(d + (\sqrt{\epsilon} - 1)b\big)^2}$$

So we can see from this equation that force of attraction changes and is smaller in this case $(d_{eff} > d)$.

Answer: when the slab is inserted the force of interaction changes (becomes smaller).

(c) An electric potential is described, in Cartesian coordinates, by $V = 3x^2 + 4y^3$. Find the electric field associated with this potential.

Solution: Knowing the electric potential V, we can find the electric field $\vec{E}$ from the equation $\vec{E} = -\nabla V$, where $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$, here $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in positive direction of x, y, z −axis. Now we can express $\vec{E}$ in terms of V:

$$\vec{E} = -\hat{x}\frac{\partial V}{\partial x} - \hat{y}\frac{\partial V}{\partial y} - \hat{z}\frac{\partial V}{\partial z}.$$

Let us find the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{\partial(3x^2 + 4y^3)}{\partial x} = 6x;$$
$$\frac{\partial V}{\partial y} = \frac{\partial(3x^2 + 4y^3)}{\partial y} = 12y^2;$$
$$\frac{\partial V}{\partial z} = \frac{\partial(3x^2 + 4y^3)}{\partial z} = 0.$$

So, finally, we obtain the electric field:

$$\vec{E} = -6x \cdot \hat{x} - 12y^2 \cdot \hat{y}.$$

Answer: $\vec{E} = -6x \cdot \hat{x} - 12y^2 \cdot \hat{y}.$

Question 3

A long, cylindrical, current-carrying conductor of radius a lies along the z axis of a cylindrical coordinate system. The current density is constant with radius, such that

$$\vec{J}(s) = K\hat{z} \ for \ s < a$$
$$\vec{J}(s) = 0 \ for \ s \geq a,$$

where s is the radial coordinate. The permeability is μ_0 everywhere.

(a)

(i) Draw a sketch of the conductor, including the coordinate system and the magnetic field lines.
Answer: we will use here cylindrical coordinate system. Magnetic field lines are coaxial circles around the z axis.

(ii) What is the direction of the magnetic field lines in terms of the standard cylindrical unit vectors?
Answer: magnetic field lines are circles. Hence the vector of magnetic induction is orthogonal to the unit vector $\hat{r}$ and has the direction of the unit vector $\hat{\theta}$.

(iii) Draw Amperian paths that could be used to determine the magnetic field inside and outside the conductor.
Answer: Amperian paths are circles centered on the z axis. One of them is denoted on the sketch with letter A.

(iv) Find the current enclosed with each path.

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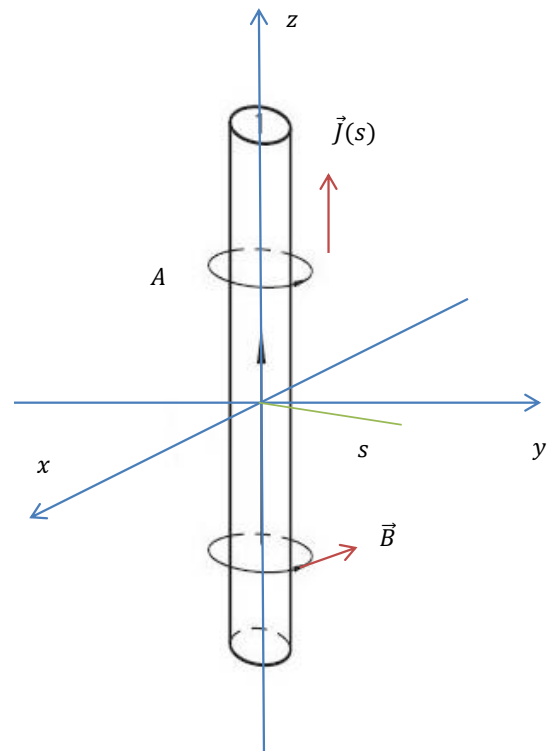
Solution: we will consider two cases: $s < a$ and $s \geq a$. For the first case current density is $\vec{J}(s) = K\hat{z}$ and for the second $\vec{J}(s) = 0$. Since the current density is constant everywhere and does not depend on radial distance s, we can simply write that the enclosed current is $I = J(s) \cdot S$, where S is area of the cross section. For $s < a$ area of the cross section is $S = \pi s^2$, so the enclosed current is $I = K\pi s^2$. For $s \geq a$ the enclosed current is $I = J(s)\pi s^2 = K \cdot \pi a^2 + 0 \cdot \pi s^2 = K\pi a^2$.

Answer: $I_{enc} = K\pi s^2$ for $s < a$, $I_{enc} = K\pi a^2$ for $s \geq a$.

(b) Use your answer to part (a) to find the magnetic field inside and outside the conductor.

Solution: let us use the Ampere's law. For $s < a$:

$$\int \vec{B} \cdot d\vec{l} = B(s) \cdot 2\pi s = \mu_0 I_{enc} = \mu_o K\pi s^2 \ \rightarrow \ B(s) = \frac{\mu_0 Ks}{2}$$

For $s \geq a$ we write

$$\int \vec{B} \cdot d\vec{l} = B(s) \cdot 2\pi s = \mu_0 I_{enc} = \mu_o K\pi a^2 \ \rightarrow \ B(s) = \frac{\mu_0 Ka^2}{2s}$$

And it is useful to remember that vector $\vec{B}$ is directed along the unit vector $\hat{\theta}$.

Answer: $\vec{B}(s) = \frac{\mu_0 Ks}{2} \hat{\theta}$ for $s < a$ and $\vec{B}(s) = \frac{\mu_0 Ka^2}{2s}\hat{\theta}$ for $s \geq a$.

Question 4

(a) A sample of crystalline cooper has a face centered cubic lattice with one atomic species in the basis.

(i)   Sketch the conventional unit cell for this structure, indicating the position of copper atoms.

Answer: typical face centered cubic lattice of crystalline cooper is represented in the picture on the right. Gray circles are the atoms of the cooper.
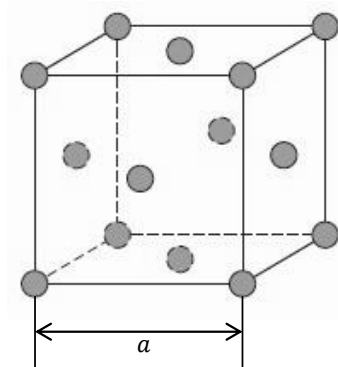
(ii)   Indicate the lattice constant a, on your sketch.

Answer: lattice constant is the constant distance between unit cells in a crystal lattice. For a cubic lattice this distance is similar in all three directions (x, y, z). On the sketch it is indicated with letter a.

(iii)   On your sketch shade the (110) plane.

Answer: to sketch this plane we should take the reciprocal values of numbers 1 1 0. Hence we obtain $1, 1, \infty$. This means that the plane parallels the z axis while intersecting both x and y axes at a. This plane is denoted with the blue lines on the sketch below.

(iv)   How many atoms are there in the (110) plane of the unit cell?

Answer: as we see from the sketch there is six atoms in the (110) plane. Four of them are located in the corners of the cubic lattice and two of them are in the centers of the faces.
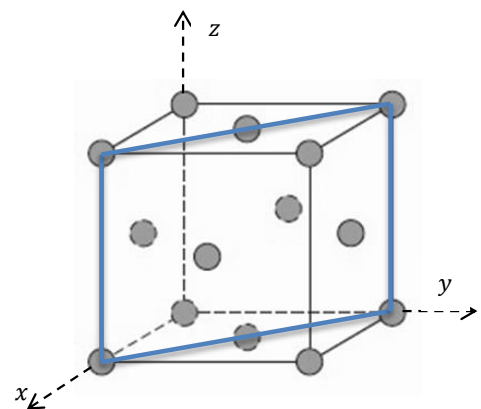
(v)   Hence calculate the density of atoms (number per unit area) for this face in terms of lattice constant.

Solution: area of this face is the area of the rectangle, the lengths of its sides are a and $a\sqrt{2}$ (diagonal of the

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square with side a). So the density is

$$\rho = \frac{Number\ of\ atoms}{Area} = \frac{6}{a^2\sqrt{2}} = \frac{3\sqrt{2}}{a^2}.$$

**Answer:** $\rho = \frac{3\sqrt{2}}{a^2}$.

(b) If the radius of the copper atom is 0.1278 nm, determine the lattice constant of FCC copper.

Solution: on the sketch is illustrated the model of the FCC cubic lattice, where atoms are represented as hard spheres with radius R that just touch each other. From this picture is easy to see that

$$a^2 + a^2 = (4R)^2$$

From this equation we obtain $a^2 = 8R^2$, so $a = 2\sqrt{2} \cdot R = 0.3615\ nm$.

Answer: $a = 2\sqrt{2} \cdot R = 0.3615\ nm$.
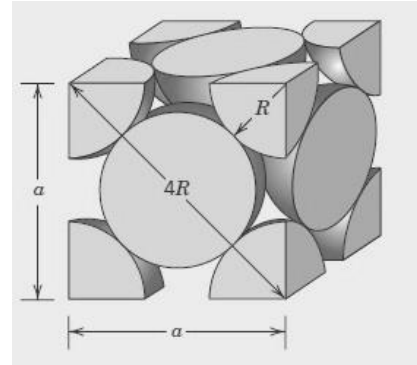
(c)

    (i)   Sketch a cube to represent the frame of the conventional unit cell, and on it sketch the directions $[\bar{1}10]$, and $[\bar{1}11]$.

            Answer: direction $[\bar{1}10]$ in terms of coordinates in this cube lattice is a vector from the origin to the point $(-a, a, 0)$. Similarly with the direction $[\bar{1}11]$: it is a vector from the origin to the point $(-a, a, a)$.
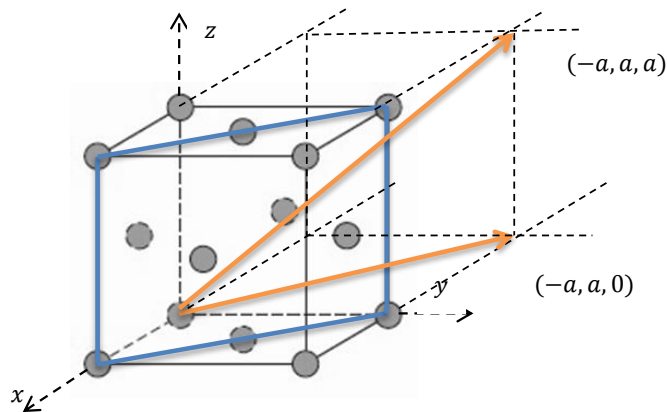
    (ii)   Prove that these directions are in (110) plane.

            Answer: through two vectors that form directions $[\bar{1}10]$ and $[\bar{1}11]$ we can pass a plane. As we can see from the sketch this plane is parallel to the plane (110. In respect to the origin at the point $(-a, 0, 0)$ (in old coordinates) this plane is also (110) plane.

Question 5

(a) Consider a pair of Mg^{2+} and O^{2-} ions that just touch each other. Their centers are separated by a distance R. The attractive force is given by $F_{attractive}(R) = \frac{q_1 q_2}{4\pi\epsilon_0 R^2}$, and the repulsive force by $F_{repulsive}(R) = \frac{-nb}{R^{n+1}}$. Useful data: $\epsilon_0 = 8.85 \cdot 10^{-12}\ \frac{C^2}{Nm^2}$, Mg^{2+} ionic radius is $r_1 = 0.065\ nm$, O^{2-} ionic radius $r_2 = 0.140\ nm$.

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(i) Determine the constant b in the expression for the repulsive force given that the repulsive force equals the attractive force at equilibrium. Assume $n = 7$.

Solution: in this model we assume that ions just touch each other. So we can write that the distance between their centers is $R = r_1 + r_2$, where r_1, r_2 the ionic radii of two ions are. Now we use the equilibrium condition:

$$F_{repulsive}(R) = F_{attractive}(R)$$

$$\frac{q_1 q_2}{4\pi\epsilon_0 R^2} = \frac{-nb}{R^{n+1}}$$

So we become:

$$b = -\frac{q_1 q_2}{4\pi\epsilon_0 n} R^{n-1}$$

In our case $q_1 = +2e$ (Mg^{2+}charge) and $q_2 = -2e$ (O^{2-} charge), $R = r_1 + r_2 = 0.205\ nm$, $n = 7$, so the formula takes form

$$b = \frac{e^2}{7\pi\epsilon_0}(r_1 + r_2)^6$$

$$b = \frac{(1.6 \cdot 10^{-19})^2}{7\pi \cdot 8.85 \cdot 10^{-12}}(0.065 + 0.140)^6 \cdot 10^{-54} = 0.98 \cdot 10^{-86}\ J \cdot m^7.$$

We can express this result in $eV \cdot nm^7$ units. Since $1\ Electronvolt = 1.6 \cdot 10^{-19}\ J$ and $1\ nm = 10^{-9}m$, so $1\ J \cdot m^7 = \frac{10^{82}}{1.6}\ eV \cdot nm^7$. In these units $b = 0.61 \cdot 10^{-4}\ eV \cdot nm^7$.

Answer: $b = \frac{e^2}{7\pi\epsilon_0}(r_1 + r_2)^6 = 0.61 \cdot 10^{-4}\ eV \cdot nm^7$.

(ii) Hence calculate the net potential energy for an $Mg^{2+}O^{2-}$ ion par by using this constant b.

Solution: Since $F = -\frac{\partial U}{\partial R}$, we are able to find the net potential energy as $U = -\int_{\infty}^{R} F dR$, here F is the total force that acts on the ion and

$$F = F_{attractive} + F_{repulsive} = \frac{q_1 q_2}{4\pi\epsilon_0 R^2} - \frac{7b}{R^8}$$

Hence the total net potential energy is

$$U = -\int_{\infty}^{R}\left(\frac{q_1 q_2}{4\pi\epsilon_0 R^2} - \frac{7b}{R^8}\right) dR = \frac{q_1 q_2}{4\pi\epsilon_0 R} + \frac{b}{R^7}$$

$$U = -\frac{4 \cdot (1.6 \cdot 10^{-19})^2}{4 \cdot 3.14 \cdot 8.85 \cdot 10^{-12} \cdot 0.205 \cdot 10^{-9}} + \frac{0.98 \cdot 10^{-86}}{(0.205)^7 \cdot 10^{-63}} = -3.85 \cdot 10^{-18}\ J.$$

Remembering that $1\ eV = 1.6 \cdot 10^{-19}\ J$, we obtain $U = -28.09 + 4.01 = -24.08\ eV$.

Answer: $U = \frac{q_1 q_2}{4\pi\epsilon_0 (r_1 + r_2)} + \frac{b}{(r_1 + r_2)^7} = -24.08\ eV$.

(iii) Is b the binding energy? Justify your answer.

Answer: binding energy is the mechanical energy required to dissemble a whole system into separate parts. This energy is used against the forces that hold the system together. In our case these forces are electrical forces, so the binding energy is the total net potential energy of the ions but with an opposite sign $E_{binding} = -U = 24.08\ eV$. Parameter b is not the binding energy, it just characterizes the structure properties of the matter.

(b) Briefly explain why and how doping a semiconductor increases the conductivity. In your explanation define the following terms:

(i) Intrinsic semiconductor.

Answer: an intrinsic semiconductor (or undoped semiconductor) is a semiconductor without any significant dopant species present. The number of charge carries is only determined by

the properties of the material itself (electrical conductivity depends on the defects of crystal lattice and of number of excited electrons). Hence in an intrinsic semiconductor the number of electrons in the conduction band is equal to the number of holes in the valence band.

(ii)   Fermi level.

Answer: the Fermi level of a body is a thermodynamic quantity that shows how much work is required to be done to add one electron to the body. From other considerations one can see that Fermi level is the highest filled energy level of an electron in the ground state of a rigid body.

(iii)   n-type doping.

Answer: Addition of 0.001% of donates to an intrinsic semiconductor is able to increase its electrical conductivity by a factor of 10000. The materials chosen as suitable dopants depend on the atomic properties of both the dopant and the material to be doped. Semiconductors doped with donor impurities are called n-type. The majority charge carriers in them are electrons. The pure semiconductor silicon has four valence electrons which bond each silicon atom to its neighbors. Group V elements have five valence electrons, which allow them to act as donors. Substitution of   these atoms into the silicon creates an extra free electron.
This process is called n-type doping.
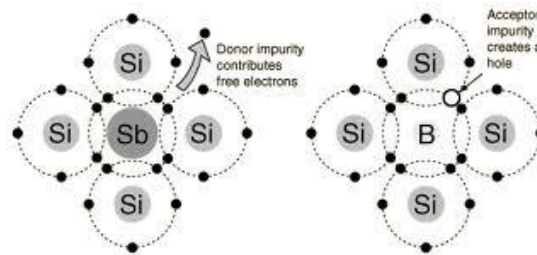
(iv)   p-type doping.

Answer: semiconductors doped with acceptor impurities are called p-type. The majority charge carriers in them are "holes". Let us come back to the silicon. Group III elements contain three valence electrons, causing them to function as acceptors when the silicon is doped by them. When acceptor atom replaces a silicon atom in the crystal, a vacant state ("hole") is created, which can move around the lattice and functions as a charge carrier. This process is called p-type doping.

Question 6

The Fe-C phase diagram is provided on an accompanying sheet. This diagram is the one used for determining the phases present in steels.

(a) An iron alloy with 1.5% C by weight is cooled from 1500 °C .

(i)   Assume that cooling occurs through equilibrium conditions. Draw lines on the diagram to show the equilibrium phases that occur in the alloy as it cools.

Answer: at the diagram on the next page is represented equilibrium cooling of an iron alloy with 1.5% C from 1500 °C . At the beginning the alloy is in the Liquid phase. Then it goes

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into Liquid + Austenite phase, Austenite phase ($\gamma - Fe$), further cooling brings the alloy into Austenite + Cementite phase and, finally, Ferrite ($\alpha - Fe$) + Cementite phase.

(ii)   At what temperature will it start to solidify?

Answer: the first solid phase that occurs in the alloy is Austenite (γ), from this diagram we can roughly estimate the temperature of the beginning of solidification (blue line). $T \approx 1455 \pm 15°C$.

(iii)   What phases will be presented when solidification begins?

Answer: at the beginning of the solidification there would be presented Liquid and Austenite (γ) phases.
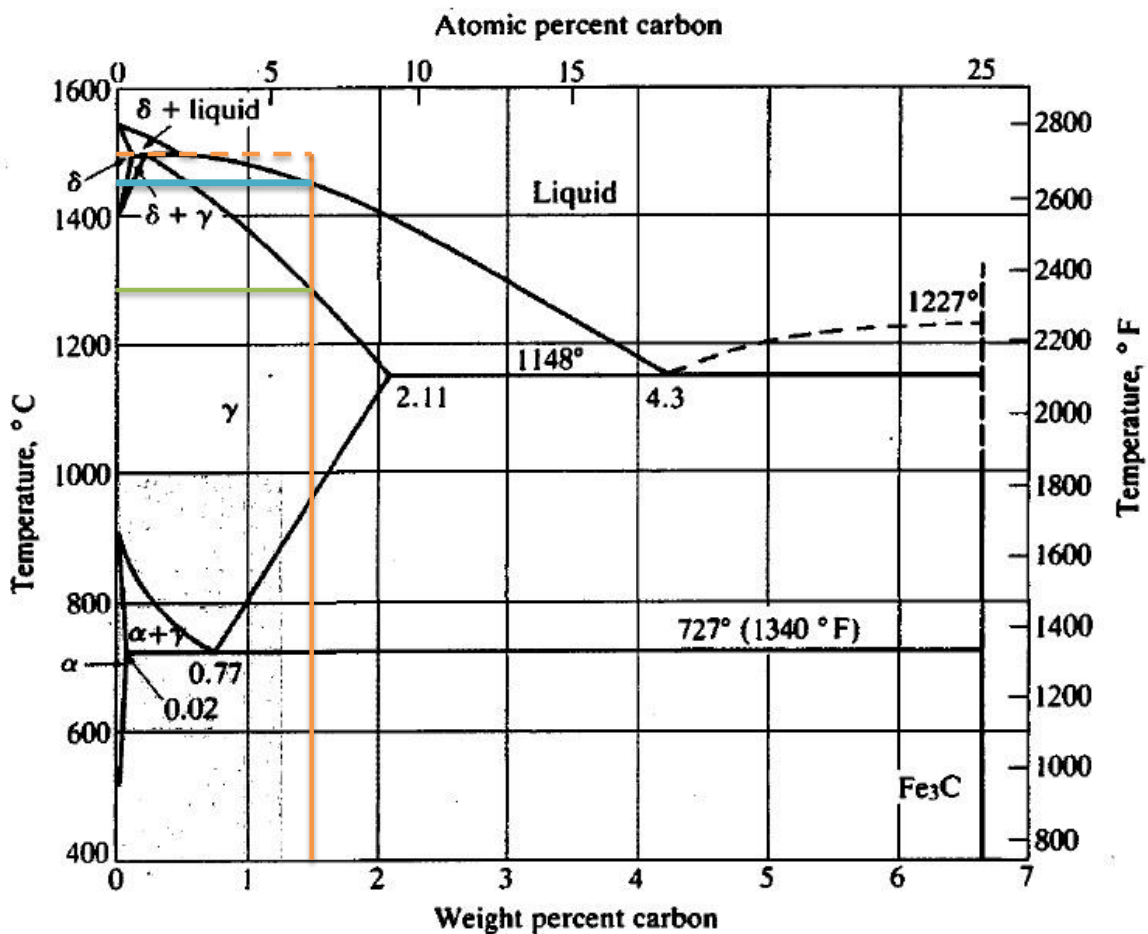
(iv)   What is the temperature at which this alloy is completely solid?

Answer: this alloy will be completely solid when the Austenite (γ) phase (green line) occurs. $T \approx 1290$ °C.

(b) What is the invariant reaction involved in this cooling and at what concentrations and temperature does it occur?

Answer: variant (b) – Eutectoid (transformations between several solid phases). It occurs in alloys containing 0.76-2.14 % C. This reaction occurs from temperatures within the γ phase.