Sample: Electrodynamics - Electromagnetic Waves

Problem 1. Establish the relation between the E-wave and H-wave amplitudes.

Solution

For vacuum ($j = 0, \rho = 0, \mu = 1, \epsilon = 1$) Maxwell equations can be written as:

$$curl\,\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$curl\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$div\,\vec{E} = 0$$

$$div\,\vec{H} = 0$$

These equations can be reduced, e.g. for $\vec{E}$ (and equivalent for $\vec{H}$)

$$curl\,curl\,\vec{E} = grad\,div\,\vec{E} - \nabla^2\,\vec{E} = -\frac{1}{c}\,curl\frac{\partial \vec{H}}{\partial t} = -\frac{1}{c}\frac{\partial\,curl\vec{H}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

So:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2\ \nabla^2 \vec{E}$$

Similarly:

$$\frac{\partial^2 \vec{H}}{\partial t^2} = c^2\ \nabla^2 \vec{H}$$

Solutions for this wave equations that describe EM waves are:

$$\vec{E}\,(\vec{r},t) = \overrightarrow{a_E}\cos\left(\overrightarrow{k_n}\cdot\vec{r} - \omega t + \delta_E\right)$$

$$\vec{H}\,(\vec{r},t) = \overrightarrow{a_H}\cos\left(\overrightarrow{k_n}\cdot\vec{r} - \omega t + \delta_H\right)$$

From Maxwell's equations follow also the relations

$$\vec{E} = -\vec{s}\times\vec{H}$$

$$\vec{H} = \vec{s}\times\vec{E}$$

$$\vec{E}\vec{s} = \vec{H}\vec{s} = 0$$

expressing that the three vectors $\vec{E}$, $\vec{H}$, and $\vec{s}$ form a right-handed orthogonal triad of vectors. Thus we can choose the z-axis in the propagation direction $s^{\rightarrow}$, so that there are only electric and magnetic field components in the x- and y-direction. The end point of the electric and magnetic vectors is then described by:

$$E_x\,(z,t) = a_x \cos(k_n \cdot z - \omega t + \delta_x)$$

$$E_y\,(z,t) = a_y\, \cos\left(k_n \cdot x - \omega t + \delta_y\right)$$

$$H_x(z,t) = E_y(z,t)$$

$$H_y(z,t) = E_x(z,t)$$

So for vacuum:

$$\boxed{\frac{E_x(z,t)}{H_y(z,t)} = 1}$$

Problem 2. Show that in a weakly conducting medium, an electromagnetic wave gets rapidly attenuated with distance.

Solution

Considering the propagation of an electromagnetic wave through a conducting medium which obeys Ohm's law:

$$\vec{j} = \sigma\vec{E}$$

Here, σ is the conductivity of the medium in question. Maxwell's equations for the wave take the form:

$$curl\,\vec{H} = \mu_0 j + \epsilon\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

$$curl\,\vec{E} = -\frac{\partial\vec{H}}{\partial t}$$

$$div\,\vec{E} = 0$$

$$div\,\vec{H} = 0$$

where ϵ is the dielectric constant of the medium. It follows, from the above equations, that

$$\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_o \sigma \vec{E} + \epsilon\epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Looking for a wave-like solution of the form

$$\vec{E} = E_o e^{i(kz-\omega t)}$$

we obtain the dispersion relation

$$k^2 = \mu_0\omega(\epsilon\epsilon_0\omega + i\,\sigma)$$

Consider a ``weak" conductor for which $\epsilon\epsilon_0\omega \gg \sigma$. In this case, the dispersion relation yields

$$k \cong n\frac{\omega}{c} + i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\epsilon\epsilon_0}}$$

Substitution into wave equation gives:

$$\vec{E} = E_o e^{-\frac{z}{d}} e^{i\omega(\frac{\sqrt{\epsilon}}{c}z - t)}$$

Where

$$\boxed{d = \frac{2}{\sigma}\sqrt{\frac{\epsilon\epsilon_0}{\mu_0}}}$$

Problem 4. What is skin effect? What is skin depth? What information and what parameters would you need to determine skin depth of a given medium?

Solution

Skin effect is the phenomenon when an alternating current tends to concentrate in the outer layer of a conductor, caused by the self-induction of the conductor and resulting in increased resistance.

As it was shown in problem 2, the dispersion relation for EM wave in conducting material is:

$$k^2 = \mu_0\omega(\epsilon\epsilon_0\omega + i\,\sigma)$$

In problem 2 we conclude that the amplitude of an electromagnetic wave propagating through a conductor decays exponentially on some length-scale, d, which is termed the skin-depth. As it is seen from problem 2 solution the skin-depth for a poor conductor is independent of the frequency of the wave.

Considering a ``good'' conductor for which $\sigma \gg \epsilon\epsilon_0\omega$. In this case, the dispersion relation yields

$$k \cong \sqrt{i\mu_0\,\sigma\omega}$$

Substitution into solution of wave equation gives:

$$\vec{E} = E_o e^{-\frac{z}{d}} e^{i\omega(\frac{\sqrt{\epsilon}}{c}z - t)}$$

Where

$$d = \sqrt{\frac{2}{\mu_0\sigma\omega}}$$

It can be seen that the skin-depth for a good conductor decreases with increasing wave frequency.

Problem 5. Concerning electromagnetic waves in a weakly conduction dielectric, prove that the B-waves lags in phase being the E-wave and find an expression for this phase difference.

Solution

As it was shown in problem 2

$$\vec{E} = E_o e^{i(kz - \omega t)}$$

and the dispersion relation

$$\boxed{k^2 = \mu_0\omega(\epsilon\epsilon_0\omega + i\,\sigma)}$$

Similarly for magnetic field:

$$\vec{B} = B_o e^{i(kz-\omega t)}$$

Using Maxwell's equations:

$$curl\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Which gives:

$$\vec{k} \times \overrightarrow{E_0} = \omega \boldsymbol{B}$$

Or

$$\overrightarrow{B_0} = \frac{\vec{k}}{\omega} \times \overrightarrow{E_0}$$

So in a conductor, the complex phase of $\vec{k}$ gives a phase difference between the electric and magnetic fields. This phase difference is given by the phase angle ϕ of $\vec{k}$

$$\boxed{\tan\phi = -\frac{Im(\vec{k})}{Re(\vec{k})}}$$

Problem 6. Calculate the time averaged energy density of an electromagnetic wave in a weakly conducting medium.

Solution

The power per unit volume dissipated via ohmic heating in a conducting medium takes the form

$$P = \mathbf{j} \cdot \mathbf{E} = \sigma\, E^2.$$

Consider an electromagnetic wave of the form

$$\vec{E} = E_o e^{-\frac{z}{d}} e^{i\omega\left(\frac{\sqrt{\epsilon}}{c} z - t\right)}$$

The mean power dissipated per unit area in the region $z > 0$ is written

$$\langle P\rangle = \frac{1}{2}\int_0^\infty \sigma\, E_0^2\, e^{-2\,z/d}\,dz = \frac{d\,\sigma}{4}\,E_0^2 = \sqrt{\frac{\sigma}{8\,\mu_0\,\omega}}\,E_0^2,$$

for a good conductor.
Now, according to equation

$$\mathbf{u} = \frac{|E|^2}{\mu_0\,\omega}\,\mathrm{Re}(\mathbf{k}).$$

the mean electromagnetic power flux into the region $z > 0$ takes the form

$$\langle u\rangle = \left\langle\frac{\mathbf{E}\times\mathbf{B}\cdot\hat{\mathbf{z}}}{\mu_0}\right\rangle_{z=0} = \frac{1}{2}\frac{E_0^2\,k_+}{\mu_0\,\omega} = \sqrt{\frac{\sigma}{8\,\mu_0\,\omega}}\,E_0^2.$$

It is clear from a comparison of the previous two equations, that all of the wave energy which flows into the region z>0 is dissipated via ohmic heating.

Problem 7. Concerning EM waves in a weakly conducting medium, find an expression for the phase velocity of the wave.

Solution

From solutions of problems above for weakly conducting medium:

$$\vec{E} = E_o e^{-\frac{z}{d}} e^{i\omega\left(\frac{\sqrt{\epsilon}}{c}z-t\right)}$$

The phase velocity is the velocity of a point that stays in phase with the wave,

For point staying at a fixed phase, we must have:

$$\omega\left(\frac{\sqrt{\epsilon}}{c}z - t\right) = const$$

$$\omega\frac{\sqrt{\epsilon}}{c}z = \omega t + const$$

So the phase velocity is given by:

$$\boxed{v_p = \frac{dz}{dt} = \frac{c}{\sqrt{\epsilon}}}$$