Sample: Integral Calculus - Directional Derivative and Total Differential

- *Find the directional derivative of the function* $z = \frac{1}{2} \ln \tan x - \sin^2 y$ *at a point* $(\frac{\pi}{4}, \frac{3\pi}{4})$ *in the direction of the vector* $v = -i + \sqrt{3}j$.

$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{\tan x}\frac{1}{\cos^2 x} = \frac{\cos x}{2\sin x}.$

$\frac{\partial z}{\partial y} = -2\sin y\cos y.$

$\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{4},\frac{3\pi}{4})} = \frac{1}{2}.$

$\left.\frac{\partial z}{\partial y}\right|_{(\frac{\pi}{4},\frac{3\pi}{4})} = 1.$

We need to find the unit $\vec{u}$ vector in the direction of $\vec{v} = (-1, \sqrt{3})$, which is

$\vec{u} = \frac{\vec{v}}{|v|} = \frac{(-1,\sqrt{3})}{\sqrt{1+3}} = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right).$

Thus, the directional derivative at $(\frac{\pi}{4}, \frac{3\pi}{4})$ in the specified direction is

$\frac{\partial z}{\partial v} = \nabla z \cdot \vec{u} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) + 1\cdot\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} - \frac{1}{4}$

Answer: $\frac{\sqrt{3}}{2} - \frac{1}{4}.$

- *Find the total differential of the function* $w(x,y,z) = \frac{1}{3}\sqrt{(x^2+y^2)^3 + x - \frac{y^2}{2}} + \tan^{-1}\frac{y}{z}$ *at the point (0, 1, 2).*

We begin by finding the partial derivatives of w:

$$w_x = \left.\frac{6x(x^2+y^2)^2+1}{6\sqrt{(x^2+y^2)^3+x-\frac{y^2}{2}}}\right|_{(0,1,2)} = \frac{\sqrt{2}}{6}.$$

$$w_y = \left.\frac{6y(x^2+y^2)^2-y}{6\sqrt{(x^2+y^2)^3+x-\frac{y^2}{2}}} + \frac{1}{(1+\frac{y^2}{z^2})z}\right|_{(0,1,2)} = \frac{5\sqrt{2}}{6} + \frac{2}{5}.$$

$$w_z = -\left.\frac{y}{(1+\frac{y^2}{z^2})z^2}\right|_{(0,1,2)} = -\frac{1}{5}.$$

Thus, we have $dw = \frac{\sqrt{2}}{6}dx + (\frac{5\sqrt{2}}{6} + \frac{2}{5})dy - \frac{1}{5}dz$

Answer: $dw = \frac{\sqrt{2}}{6}dx + (\frac{5\sqrt{2}}{6} + \frac{2}{5})dy - \frac{1}{5}dz$