Sample: Multivariable Calculus - Derivatives

1)

Let's substitute

$$u(x,t) = 2k^2 \operatorname{sech}^2\big(k(x-4k^2t)\big)$$

into the equation

$$u_t + 6uu_x + u_{xxx} = 0$$

$$u_t = 2k^2 \cdot 2\operatorname{sech}\big(k(x-4k^2t)\big) \cdot \tanh\big(k(x-4k^2t)\big)\big(-\operatorname{sech}\big(k(x-4k^2t)\big)\big)(-4k^3) = 16k^6 \operatorname{sech}^2\big(k(x-4k^2t)\big) \cdot \tanh\big(k(x-4k^2t)\big)$$

$$u_x = 2k^2 \cdot 2\operatorname{sech}\big(k(x-4k^2t)\big) \cdot \tanh\big(k(x-4k^2t)\big)\big(-\operatorname{sech}\big(k(x-4k^2t)\big)\big) \cdot k = -4k^3 \operatorname{sech}^2\big(k(x-4k^2t)\big)\tanh\big(k(x-4k^2t)\big)$$

$$u_{xxx} = 16k^5 \tanh\big(k(x-4k^2t)\big)\operatorname{sech}^2\big(k(x-4k^2t)\big)\big(2\operatorname{sech}^2\big(k(x-4k^2t)\big) - \tanh^2\big(k(x-4k^2t)\big)\big)$$

After substituting derivatives into the equation we get:

$$u_t + 6uu_x + u_{xxx} = 0$$

So $u(x,t)$ satisfies the equation.

2)

$$T(x,y,z) = x^2 - y^2 + z^2 + xz^2$$

Let's find the parameterization of the curve.

$$z = t$$

$$3x^2 - y^2 = t$$

$$2x^2 + 2y^2 = t^2$$

$$x^2 = \frac{1}{8}(t^2 + t)$$

$$y^2 = 3x^2 - t = \frac{1}{8}(3t^2 - 5t)$$

So

$$(x, y, z) = \left(\frac{\sqrt{t^2 + t}}{2\sqrt{2}}, \frac{\sqrt{3t^2 - 5t}}{2\sqrt{2}}, t\right)$$

Let's change parametrization so speed at $t = 0$ is v.

$$|x'^2 + y'^2 + z'^2|^2 = \left(\frac{2t + 1}{4\sqrt{2}\sqrt{t^2 + t}}\right)^2 + \left(\frac{6t - 5}{4\sqrt{2}\sqrt{3t^2 - 5t}}\right)^2 + 1$$
$$= \frac{(2t + 1)^2}{32(t^2 + t)} + \frac{(6t - 5)^2}{32(3t^2 - 5t)} + 1$$

AT $t = 0$ we have:

$$speed =$$

At point (1,1,2):

$$z' = 6xx' - 2yy'$$

$$4xx' + 4yy' - 2zz' = 0$$

Substituting $x = 1; y = 1; z = 2$ we have:

$$z' = 6x' - 2y'$$

$$4x' + 4y' - 4z' = 0$$

Solving this with $x'^2 + y'^2 + z'^2 = v^2$ we have:

$$x'(0) = \frac{3v}{7\sqrt{2}}; y'(0) = \frac{5v}{7\sqrt{2}}; z'(0) = \frac{8v}{7\sqrt{2}}$$

Then rate of change of temperature at $t = 0$ is

$$2x\cdot x'(0)-2yy'(0)+2zz'(0)+x'(0)z^2+xzz'(0)$$
$$=2\cdot 1\cdot\frac{3v}{7\sqrt{2}}-2\cdot 1\cdot\frac{5v}{7\sqrt{2}}+2\cdot 2\cdot\frac{8v}{7\sqrt{2}}+\frac{3v}{7\sqrt{2}}\cdot 2^2+1\cdot 2\cdot\frac{8v}{7\sqrt{2}}$$
$$=\frac{56v}{7\sqrt{2}}=4\sqrt{2}v$$

3)

Let's compute derivatives of u by new variables.

$$\frac{du}{dr}=\frac{du}{dx}\frac{dx}{dr}+\frac{du}{dy}\frac{dy}{dr}=\cos\theta\frac{du}{dx}+\sin\theta\frac{du}{dy}$$

$$\frac{d^2u}{dr^2}=\cos^2\theta\frac{d^2u}{dx^2}+2\cos\theta\sin\theta\frac{d^2u}{dxdy}+\sin^2\theta\frac{d^2u}{dy^2}$$

$$\frac{du}{d\theta}=-r\sin\theta\frac{du}{dx}+r\cos\theta\frac{du}{dy}$$

$$\frac{d^2u}{d\theta^2}=-r\left(\cos\theta\frac{du}{dx}+\sin\theta\frac{du}{dy}\right)+r^2\left(\sin^2\theta\frac{d^2u}{dx^2}-2\cos\theta\, sin\theta\frac{d^2u}{dxdy}+\cos^2\theta\frac{d^2u}{dy^2}\right)$$

Dividing both sides by r^2 and using expression for $\frac{du}{dr}$ we have:

$$\frac{1}{r^2}\frac{d^2u}{d\theta^2}=-\frac{1}{r}\frac{du}{dr}+\sin^2\theta\frac{d^2u}{dx^2}-2\cos\theta\sin\theta\frac{d^2u}{dxdy}+\cos^2\theta\frac{d^2u}{dy^2}$$

Adding last 2 relation we have:

$$\frac{d^2u}{dr^2}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}=-\frac{1}{r}\frac{du}{dr}+\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}$$

So

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}+\frac{1}{r}\frac{du}{dr}$$

So Laplace equation becomes:

$$\frac{d^2u}{dr^2}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}+\frac{1}{r}\frac{du}{dr}=0$$

4)

(a)

Operator F is defined on matrices which have the inverse.

$$dom\ F=\{(x_1,x_2,x_3,x_4)|x_1x_4\neq x_2x_3\}$$

(b)

Let's write F explicitly:

$$F(x_1,x_2,x_3,x_4)=\frac{1}{x_1x_4-x_2x_3}\begin{pmatrix}x_4 & -x_2\\ -x_3 & x_1\end{pmatrix}=$$

$$\left(\frac{x_4}{x_1x_4-x_2x_3},-\frac{x_2}{x_1x_4-x_2x_3},-\frac{x_3}{x_1x_4-x_2x_3},\frac{x_1}{x_1x_4-x_2x_3}\right)$$

Matrix of derivatives:

$$\frac{dF}{dx}=-\frac{1}{(x_1x_4-x_2x_3)^2}\cdot$$

$$\cdot\begin{pmatrix}x_4\cdot x_4 & x_4\cdot(-x_3) & x_4\cdot(-x_2) & -(x_1x_4-x_2x_3)+x_4x_1\\ -x_2x_4 & (x_1x_4-x_2x_3)-(-x_2x_3) & -x_2(-x_2) & -x_2x_1\\ -x_3x_4 & -x_3(-x_3) & (x_1x_4-x_2x_3)-x_3(-x_2) & -x_3x_1\\ -(x_1x_4-x_2x_3)+x_1x_4 & x_1(-x_3) & x_1(-x_2) & x_1\cdot x_1\end{pmatrix}$$

$$=-\frac{1}{(x_1x_4-x_2x_3)^2}\begin{pmatrix}x_4^2 & -x_3x_4 & -x_2x_4 & x_2x_3\\ -x_2x_4 & x_1x_4 & x_2^2 & -x_2x_1\\ -x_3x_4 & x_3^2 & x_1x_4 & -x_3x_1\\ x_2x_3 & -x_1x_3 & -x_1x_2 & x_1^2\end{pmatrix}$$