Sample: Complex Analysis - Cauchy Integral Formula

Task 1

Find the domain of analyticity of the function $f(z)=1/(z-3)+\cos^2 z$.

Solution.

As function $\cos z$ is analytic everywhere in C, so the domain of analyticity of the function $f(z)=1/(z-3)+\cos^2 z$ is the same as for the function $g(z) = 1/(z-3)$. This function $g(z)$ is analytic in $C\setminus\{z=3\}$. So, $f(z)$ has domain of analyticity C\{z=3}.

Task 2

Use Cauchy's Integral Formula to evaluate the following integrals:

a) $\int_\Gamma \frac{e^{z^3}}{(3z-i)^2}dz$, where Γ is a circle of radius 5 centered at 4 and traversed once in the negative (with respect to the disk) direction.

b) $\int_\Gamma \frac{1}{z^3(z-2)^2}dz$, where Γ is a circle of radius 4 centered at -2 + i and traversed once in the positive (with respect to the disk) direction.

Solution

a) $z_0 = \frac{i}{3}$ - pole of order 2.

Let us find out whether $z_0 = \frac{i}{3}$ is in the interior of this circle (mark this disk as D: $D=\{z \mid |z-4|\le 5\}$):

$|\frac{i}{3}-4|\le 5$, (because $\left|\frac{i}{3}-4\right| = \sqrt{\frac{1}{9}+16} = 4.01386 < 5$) accordingly $z_0 = \frac{i}{3}$ is in the domain D.

Suppose $f(x)=e^{z^3}$, then $f'(z)=3z^2e^{z^3}$ and

$$I=\int_\Gamma \frac{e^{z^3}}{(3z-i)^2}dz=\frac{1}{9}\int_\Gamma \frac{f(z)}{(z-\frac{i}{3})^2}dz=\frac{1}{9}\frac{2\pi i}{1!}\left(\frac{1!}{2\pi i}\cdot\int_\Gamma \frac{f(z)}{(z-\frac{i}{3})^2}dz\right)$$

Using Cauchy's Integral Formula we obtain

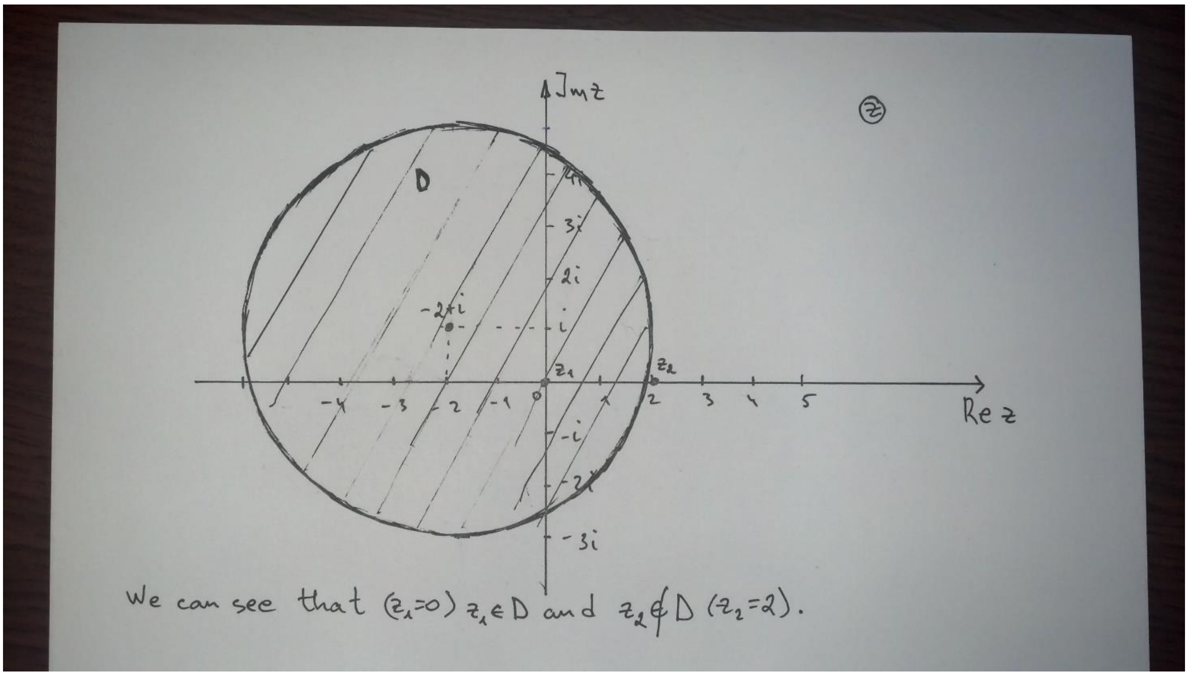
$$I=\frac{1}{9}\frac{2\pi i}{1}\cdot f'(\frac{i}{3})=\frac{2}{9}\pi i\cdot(e^{z^3})'\,|_{z=\frac{i}{3}}=\frac{2\pi i}{9}e^{z^3}\cdot 3z^2\,|_{z=\frac{i}{3}}=-\frac{2i\pi}{27}e^{-\frac{i}{27}}.$$

b) Let D is the domain that is located in the interior of the curve Γ: $D=\{z \mid |z+2-i|\le 4\}$.

Function has such singular points:

1) $z_1=0$ - the third order pole.

2) $z_2=2$ - the second order pole.



Introduce the notation $\phi(z) = \dfrac{1}{(z-2)^2}$

Then

$$I = \int_\Gamma f(z)dz = \int_D \frac{\phi(z)}{(z-0)^3}dz = \frac{2\pi i}{2!}\left(\frac{2!}{2\pi i}\int_D \frac{\phi(z)}{(z-0)^3}dz\right) =$$

{since $\phi(z)$ is analytic in D, then using Cauchy's Integral Formula we obtain:}

$$= \frac{2\pi i}{2!}\cdot\phi''(z)|_{z=0} = \pi i\cdot\left(-\frac{2}{(z-2)^3}\right)|_{z=0} = \pi i\left(\frac{6}{(z-2)^4}\right)|_{z=0} = \frac{6\pi i}{16} = \frac{3\pi i}{8}.$$

Task 3

Let f be an entire function. Assume that $|f(z)| \le 2|e^{z^2}|$ for all z in C and such that f(0) = 1. Prove that $f(z) = e^{z^2}$ for all z in C.

Solution

If f is entire function then $g(z) = e^{-z^2}f(z)$ is entire too. From $|f(z)| \le 2|e^{z^2}|$, we obtain $|g(z)| = \left|e^{-z^2}f(z)\right| \le \left|e^{-z^2}2e^{z^2}\right| = 2$. It means g is bounded and from Liouville's theorem we get that g is constant. From condition f(0)=1, we get $g(z) = e^0 * f(0) = 1$ in C. Thus, $f(z) = e^{z^2}$.

Task 4

Find the Taylor series expansion of the given analytic function f(z), centered at point z_0; find the disk of convergence:

a) $f(z) = \frac{1}{-1+4i-z}, \quad z_0 = 2.$

b) $f(z) = (z^2 - 3)e^{z^2}, \quad z_0 = 0.$

Solution

a) $f(z) = \frac{1}{4i-1-z}, \quad z_0 = 2.$

Let $z-2=t$, then

$$f(z) = f(t+2) = \frac{1}{4i-3-t} = \frac{1}{4i-3} * \frac{1}{1-\frac{t}{4i-3}}$$

Notice, that

$$\frac{1}{1-k} = \sum_{n=0}^{\infty} k^n \quad |k|<1.$$

The formula above is the sum of infinitely decreasing progression. Hence, when $|\frac{t}{4i-3}|<1$, we get

$$f(z) = \frac{1}{4i-3}\sum_{n=0}^{\infty}(\frac{t}{4i-3})^n$$

Since $t = z-2$: $\left|\frac{t}{4i-3}\right| < 1$, so

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-2)^n}{(4i-3)^{n+1}}.$$

The disk of convergence for this extension is

$$D = \{z \in C : |z-2| < |2+1-4i| = 5\},$$

b) $f(z) = (z^2-3)e^{z^2}$, $z_0 = 0.$

It is known, that

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!},$$

Hence,

$$e^{z^2} = \sum_{n=0}^{\infty}\frac{z^{2n}}{n!}.$$

Thus,

$$f(z) = (z^2-3)\sum_{n=0}^{\infty}\frac{z^{2n}}{n!} = \sum_{n=0}^{\infty}\frac{z^{2n+2}}{n!} - \sum_{n=0}^{\infty}\frac{3z^{2n}}{n!}$$

Since, $\sum_{n=0}^{\infty}\frac{z^{2n+2}}{n!} = \sum_{n=1}^{\infty}\frac{z^{2n}}{(n-1)!}$ we obtain

$$f(z) = \sum_{n=1}^{\infty}\frac{z^{2n}}{(n-1)!} - \sum_{n=0}^{\infty}\frac{3z^{2n}}{n!} = 3 + \sum_{n=1}^{\infty}\frac{z^{2n}}{(n-1)!} - \sum_{n=1}^{\infty}\frac{3z^{2n}}{n!} = 3 + \sum_{n=1}^{\infty}(\frac{z^{2n}}{(n-1)!} - \frac{3z^{2n}}{n!}) =$$

$$= 3 + \sum_{n=1}^{\infty} z^{2n}(\frac{1}{(n-1)!} - \frac{3}{n!}) = 3 + \sum_{n=1}^{\infty} z^{2n}\frac{n-3}{n!}$$

This function is analytic in C, so such expansion converges for every disk in C. (By the ratio test, the series converges).

Task 5

Find the Taylor series expansion (centered at $z_0 = 0$) of the function $f(z) = z\sin(z^2)$. Use this expansion: a) to find $f^{(103)}(0)$; b) to compute the integral traversed once in the positive (with respect to the disk) direction

$$\oint_{|z|=2} \frac{f(z)}{z^{103}} dz.$$

Solution.

$f(z) = z\cdot\sin z^2$.

It is well known, that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

When $x = z^2$ we get

$\sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!}$.

So,

$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+3}}{(2n+1)!}$.

Note $4n+3 = k$, then $n = \frac{k-3}{4}$. Rewrite our series using k (note that the summation is only over those $k \in N$ for which $\frac{k-3}{4} \in N$:

$$f(z) = \sum_{k=3,\,(k-3)\vdots 4}^{\infty} \frac{z^k (-1)^{\frac{k-3}{4}}}{(\frac{k-1}{2})!},$$

(notice: 1) $2n+1 = 2\left(\frac{k-3}{4}\right) + 1 = \frac{k-1}{2}$; 2) *we can write* : k=3 and (k-3)⋮4" or $\frac{k-3}{4} \in N$" *in the sum*). $(k-3) \vdots 4$ *means that* $\frac{k-3}{4} = r \in N$.

That is $f(z) = \sum_{k=3,\,(k-3)\vdots 4}^{\infty} a_k z^k$,

where

$a_k = \frac{(-1)^{\frac{k-3}{4}}}{(\frac{k-1}{2})!}$ (*)

Let's find $f^{(103)}(0)$. It is a_{103} term in (*) multiply on k!.

So, we obtain: $f^{(103)}(0) = 103! * a_{103} = 103! * \frac{(-1)^{25}}{\left(\frac{103-1}{2}\right)!} = 103! * \frac{-1}{51!} = -\frac{103!}{51!}$.

Evaluate: $\int_{|z|=2} \frac{f(z)}{z^{103}} dz$.

$$\int_{|z|=2} \frac{f(z)}{z^{103}} dz = \int_{|z|=2} \frac{\sum_{k=3,\,(k-3)\vdots 4}^{\infty} a_k z^k}{z^{103}} dz =$$

$$= \sum_{k=3,\,(k-3)\vdots 4}^{99} \int_{|z|=2} \frac{a_k}{z^{103-k}} dz + \int_{|z|=2} a_{103} dz + \sum_{k=107,\,(k-3)\vdots 4}^{\infty} \int_{|z|=2} a_k z^{k-103} dz \qquad (1)$$

The last two terms equal zero because it is integrals of analytic functions in the area of integration. Therefore, there is only the first term.

In fact, it is also equal to zero. Recall the result of the integral Cauchy theorem:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_D \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad z_0 \in D. \qquad (2)$$

Consider at most k for k<103. The largest such index is $k = 99$. Then:

$$I_{99} = \int_{|z|=2} \frac{a_{99}}{z^4} dz$$

It is clear that this integral is zero because according to (2) $I_{99} = \frac{2\pi i}{3!} \cdot (a_k)^{(3)} \equiv 0$ .A similar situation is with integrals in the case when k <99 ($I_{95,...}$). Therefore, we conclude that all terms in (1) are equal to 0.